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PLANE ALGEBRAIC CURVES

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PLANE ALGEBRAIC CURVES

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PREFACE

THOUGH the theory of plane algebraic curves still attracts mathematical students, the English reader has not many suitable books at his disposal. Salmon's classic treatise supplied all that could be desired at the time of its appearance, but the last edition was published some forty years ago, and has been long out of print. It seemed therefore as if a new book on the subject might be useful, if only to bring some more recent developments within reach of the student.

In the preparation of this volume I have made frequent use of the books written by Salmon, Basset, Wieleitner, Teixeira, Loria, &c. But most of the contents and examples are extracted from a very large number of mathematical periodicals. With the exception of the list at the end of Ch. XX, I have not attempted to give systematic references. In fact, in a field which has attracted so many workers, it would be almost impossible to trace the steps by which particular results have reached their present form. In some cases I cannot even remember whether a result is my own or not; but Chapters IX, XI, XVII, and XVIII contain most of my own contributions to the subject. The solutions are mine for the most part, even in the case of examples derived from other authors.

In a book dealing with so wide a subject I can hardly hope to escape the criticism that I have included just that material which happens to interest myself, and have excluded other matter of equal or greater importance. I have not seriously dealt with problems of enumeration, such as 'How many conics touch five given conics?' I have treated all curves with the same degree and singularities as forming a single type, and have not attempted to subdivide the type by considering all their possible positions relative to the line at infinity. I have not given the properties of 'special plane curves', unless they are representative of some general type, such as, for example, Cassinian curves, into which any quartic with two unreal biflecnodes can be projected. I have not included any discussion of curves of degree n for special values of n other than 2, 3, or 4. A thorough discussion of quintic curves would be very welcome, but at present the difficulties seem insuperable. At any rate very little work has been published on their properties. The reader will doubtless detect other important omissions. But on the whole I have tried to cover the limited ground I have selected with reasonable completeness.

No one can really master a branch of mathematics except by working at it himself. I make no apology, therefore, for the long lists of examples. The reader can select from them few or many, as he pleases. I give hints for solution in most cases. I hope that these will be of real assistance to the student, setting him on the right track if he is in difficulties, enabling him to check the accuracy of his results, and giving him a guarantee that the examples are not of unreasonable difficulty.

The beginner should not attempt to read the book straight through, but should select for himself those parts which he thinks easiest and likely to interest him most. As a rough guide I recommend the omission of the following portions on a first reading: Ch. VI; Ch. VII, §§ 8 to 10; Ch. VIII, §§ 4 and 5; Ch. IX, §§ 3 to 12; Ch. X, §§ 7 and 8; Ch. XI; Ch. XII, §§ 7 to 10; Ch. XVI; Ch. XVII, §§ 6 to 8; Ch. XVIII, §§ 9 to 15; Ch. XIX, §§ 3 to 7; Ch. XX, §§ 10 and 11; Ch. XXI.

My best thanks are due to friends and pupils who have made suggestions and pointed out inaccuracies while the book was being written. I owe a special debt of gratitude to Miss G. D. Sadd, who has given me very valuable lists of corrections required in the MS.

I must also express my gratitude to the Delegates of the University Press for so kindly undertaking the publication of the book.

H. H.

JUNE 1919.

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ERRATA, ETC.

Preface. While this book was in the press a treatise by S. Ganguli called *Lectures on the Theory of Plane Curves* has appeared.

Page 9, Ex. 2. Omit $(-1)^n$.

Page 10, Ex 8. Read '$2n-2$' for '$n-3$', and '$x_{2n+1}, y_{2n+1}, z_{2n+1}$' for '$x_{3n+1}, y_{3n+1}, z_{3n+1}$'.

Page 24, Ex. 12. Read 'three real tangents' for 'three tangents'. For the solution read 'Use Ex. 11 or Ch. I, § 6, Ex. 2'.

Page 116. In the theorem at the foot of the page a linear branch is counted as being superlinear of order 1.

Page 146. In Ch. X, § 4, N must be less than n. Otherwise the statement on page 146, line 33, 'By choosing t properly, P may be made any point of the n-ic' might be incorrect. Similarly in Ch. X, § 7.

Page 359, line 24. The quartic consecutive to Q is supposed non-cuspidal.

Page 369. In Fig. 13 it is essential that the nest should lie inside the circle and outside Ω.

Page 371. Add 'Göttingen Nachrichten, xi (1909), p. 308', to the list of references.

CHAPTER I

INTRODUCTORY

§ 1. Coordinates.

WE shall assume a knowledge of the more elementary portions of the calculus and of pure and analytical geometry including the theory of cross-ratio, involution, projection, reciprocation, and inversion; but in this introductory chapter we shall remind the reader of some elementary results of which we shall make frequent use.

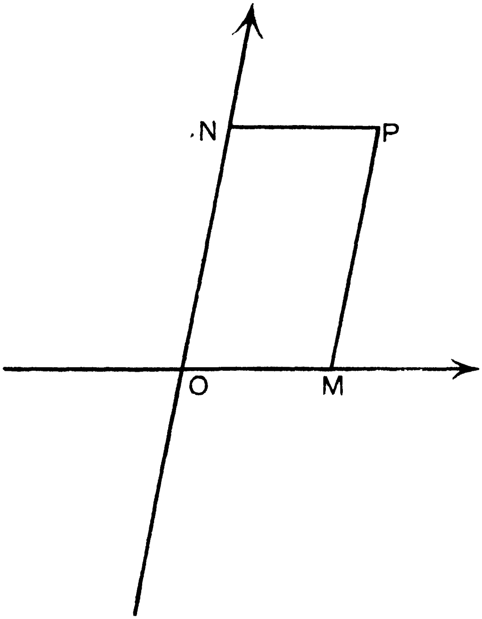

Fig. 1.

If through a point P (Fig. 1) we draw parallels PN, PM to two fixed reference lines meeting at an origin O, and OM, ON contain x and y units of length, x and y are the *Cartesian coordinates* of the point P (x, y).

If the reference-axes are perpendicular,

$$x = r\cos\theta, \quad y = r\sin\theta\,;$$

where r is the distance OP and θ is the angle between OP and $y = 0$. We call r and θ the *polar coordinates* of the point $P(r, \theta)$ with respect to the pole O and prime vector $y = 0$.

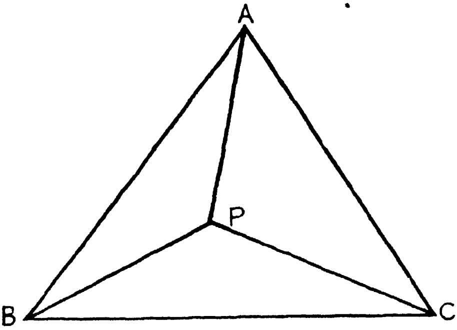

Fig. 2.

If ABC is a fixed triangle of reference (Fig. 2), the ratios x, y, z of the triangles PBC, PCA, PAB to the triangle ABC are called the *areal coordinates* of P. They are evidently connected by the relation

$$x + y + z = 1.$$

Instead of defining the position of P by its areal coordinates, we may use instead any constant multiples of them and write

$$x = k_1 . PBC/ABC, \; y = k_2 . PCA/ABC, \; z = k_3 . PAB/ABC.$$

In this k_1, k_2, k_3 are any constants chosen arbitrarily, but considered fixed when once chosen. We call x, y, z in this case general *homogeneous coordinates*. We have the relation

$$x/k_1 + y/k_2 + z/k_3 = 1,$$

by means of which the equation of any locus may be made homogeneous.

Instead of x, y, z we may take any quantities proportional to them when dealing with such homogeneous equations. For instance, the vertices of the triangle of reference will usually be taken as (1, 0, 0), (0, 1, 0), (0, 0, 1); though these are the actual coordinates only if areal coordinates are used.

We shall mean by the symbols x, y, z the general homogeneous (not necessarily the areal) coordinates, or quantities proportional to them, unless the contrary is stated.

The new equation of a given curve when fresh homogeneous coordinates (and triangle of reference) are taken is obtained by replacing x, y, z in the original equation by expressions of the form

$$l_1 x + m_1 y + n_1 z, \quad l_2 x + m_2 y + n_2 z, \quad l_3 x + m_3 y + n_3 z$$

respectively.

§ 2 Projection.

Suppose we have fixed planes Π, Π' and a fixed point V. Suppose also that P is any point in Π and that VP meets Π' in P'. Then P' is called the *projection* of P on Π', V being the 'vertex of projection'. If P traces out a locus c, P' traces out some locus c', which is called the 'projection' of c. Similarly c is the projection of c' on Π.

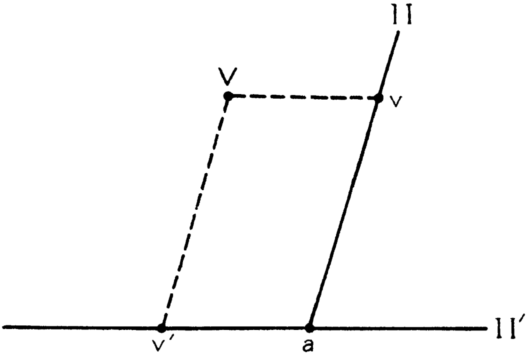

Fig. 3.

The projection of a straight line is evidently a straight line.

The projection of a range $(ABCD)$ of four collinear points whose cross-ratio is $AB.CD/AD.CB$ is a range of the same cross-ratio. For the range and its projection are the intersections of two transversals with the rays of a pencil whose vertex is V.

In particular, if the range (AC, BD) is harmonic, so that $AB.CD/AD.CB = -1$, its projection is harmonic.

Similarly the projection of a pencil of four concurrent lines is a pencil of the same cross-ratio. In fact, the pencil and its projection have the intersection a of the planes Π and Π' as a common transversal.

It follows that the projection of the 'involution range' traced out by points P, P' such that (IJ, PP') is harmonic, where I, J are the fixed 'double points' of the involution, is an involution range. Similarly for involution pencils.

Suppose that the planes through V parallel to Π' and Π meet Π and Π' in the lines v and v' respectively (see Fig. 3, in which the plane of the diagram is perpendicular to the lines v, v', a). If the point P, moving in the plane Π, approaches v, its projection P' recedes indefinitely; while if P lies on v, VP does not meet Π'; so that P has no projection.

It is convenient to observe the convention that two planes meet in a straight line, or (what is the same thing) that the projection of a straight line is a straight line, even in the case in which the planes are parallel. We say, therefore, that 'all points at infinity in the plane Π' lie on the straight line which is the projection of the *vanishing line* v'.

Similarly such a statement as 'a parabola touches the line at infinity' means that its projection touches the vanishing line; and so on. In general, when we describe any property of a curve at 'an infinitely distant point P'', we mean a property possessed by its projection at the corresponding point P of v; it being understood that the property is one which would be unaltered by projection, if P and P' were finite points. Similar remarks apply to the line v'.

Ex. 1. The relation between pole and polar with respect to a conic and between conjugate points or lines is unaltered by projection.

Ex. 2. If two lines through P in the plane Π meet v in H and K, the angle HPK projects into an angle equal to HVK.

Ex. 3. It is possible with a given vanishing line to project two angles into angles of given magnitude; or to project a given conic into a circle.

[See Ex. 1, 2. Project any two conjugate pairs of lines through the pole of the vanishing line into perpendicular pairs.]

§ 3 Plane Perspective.

Suppose that P and P', Q and Q', R and R' are projections of each other, and that Q, R are taken as fixed points of Π, and P as any other point of Π. Since $Q'R'$ is the projection of QR, QR and $Q'R'$ meet on a.

If the plane Π is turned about a carrying P, Q, R with it, while Π' is kept fixed, QR and $Q'R'$ continue to meet on a, and Q, R, Q', R' continue to be coplanar. Hence QQ' and RR' continue to intersect. Similarly PP' and QQ', PP' and RR' continue to intersect. This is only possible if PP', QQ', RR' are concurrent, since they are not coplanar. But P is any point whatever of Π. Hence when Π is rotated about a, two figures in Π and Π' which were originally projections of each other remain projections of each other. The vertex V

turns about the line v', as is evident from Fig. 3, since the distance between a and v is constant and is equal to the distance between V and v'.

Suppose that Π turns about a till it coincides with Π'. Two figures which originally were projections of one another are now two figures in the *same* plane with the property that the line joining corresponding points of the two figures passes through a fixed point V, and corresponding lines of the two figures intersect on a fixed line a. Two such figures are said to be in *plane perspective*, V being the *vertex* and a the *axis* of perspective.

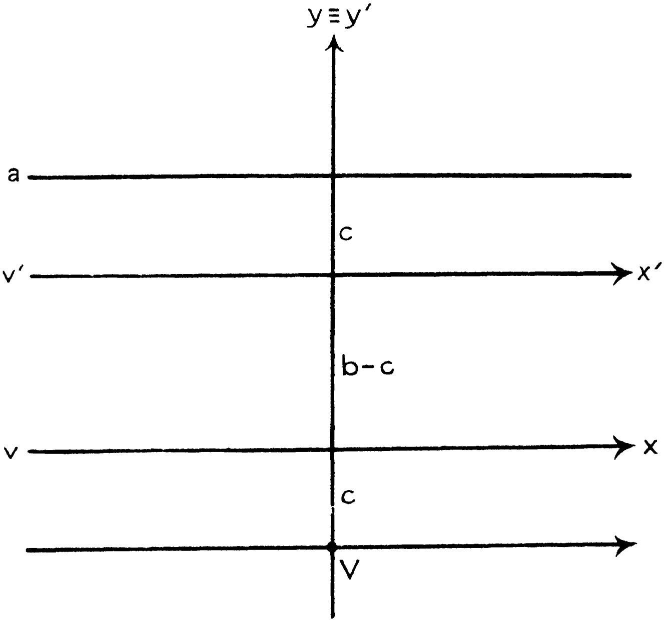

Fig. 4.

Suppose that in each of the figures the vanishing line is chosen as axis of x and the perpendicular line through V as axis of y. Suppose that $P(x, y)$ and $P'(x', y')$ are the coordinates of two corresponding points. Let c be the distance between V and v and b that between V and v', which is the same as that between v and a by Fig. 3.

Then $x' = bx/y$, $y' = bc/y$ or $x = cx'/y'$, $y = bc/y'$. . (i)

For, taking as axes of reference the lines through V parallel and perpendicular to v, the points $H(0, c)$ and $H'(0, \infty)$ correspond, while P is $(x, y+c)$. Then P' is the intersection

$(bx/y,\ b(y+c)/y)$ of the line VP and the line joining H' to the intersection of PH with the axis of perspective. But by definition P' is $(x', y'+b)$; from which (i) at once follows.

If $f(x, y) = 0$ is the equation of the locus of P,

$$f(cx/y,\ bc/y) = 0$$

is the equation of its projection.

We can always choose the projection so that $b = c$. We thus get the useful rule

If in the equation of a curve we replace x, y by ax/y, a^2/y, we get the equation of its projection; the axis of x being the vanishing line both for the curve and its projection, and the axis of y being unaltered.

Let $lx+my+n = 0$ be any line in the first figure. The distance d from it of the point $P\ (x, y)$ is given by

$$d\,(l^2+m^2)^{\frac{1}{2}} = lx+my+n.$$

The distance d' from the corresponding line in the second figure $clx+ny+bcm = 0$ of the point $P'\ (bx/y.\ bc/y)$ is given by

$$d'y\,(c^2l^2+n^2)^{\frac{1}{2}} = bc\,(lx+my+n).$$

Hence $$d'\,(c^2l^2+n^2)^{\frac{1}{2}} = bcd\,(l^2+m^2)^{\frac{1}{2}}/y. \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

Suppose that we take a fixed triangle ABC of which the vertex C coincides with V, while A and B lie on v. Then putting $n = cm$ in (ii) we see that the perpendicular from P' on CB bears a constant ratio to the quotient of the perpendiculars from P on CB and AB; and similarly for the perpendicular from P' on CA.

Now (changing the notation) we may take homogeneous coordinates (x, y, z) of P with ABC as triangle of reference, such that x/z is equal to the quotient of the perpendiculars from P on CB and AB multiplied by any constant we please; and so for y/z.

Also the perpendiculars from P' on CB and CA are constant multiples of the Cartesian coordinates of P' referred to CA and CB as axes of reference.

If, then, $f(x, y) = 0$ is the Cartesian equation of a curve, $f(x/z, y/z) = 0$ is a homogeneous equation of a projection in which $z = 0$ is the vanishing line, and $y = 0$, $x = 0$ are the projections of the Cartesian axes of reference.

Conversely, if $f(x, y, z) = 0$ is the homogeneous equation of a curve, $f(x, y, 1) = 0$ is the Cartesian equation of the curve obtained by a projection in which $z = 0$ is vanishing line, followed by an orthogonal projection.

Ex. 1. If P and P' are any two corresponding points of two figures in plane perspective and VPP' meets a in R, the cross-ratio of $(VPRP')$ is constant.

[Project a to infinity, and the figures have V as a centre of similitude. If the cross-ratio is -1, the perspective is called *harmonic*.]

Ex. 2. Given V, a, and a pair of corresponding points Q and Q', construct P' corresponding to a given point P.

[P' lies on VP and on the line joining Q' to the intersection of PQ and a.]

Ex. 3. Given V, v, a, construct P' corresponding to a given point P.

[If Q is on v, Q' is at infinity on VQ. Now use Ex. 2.]

Ex. 4. The relations connecting the coordinates of any two corresponding points of two planes which have a one-to-one correspondence can be put in the form (i) of § 3 by a suitable choice of axes of reference.

[Let I, J be the points in one plane corresponding to the circular points of the other plane (§ 5). The axes of reference are IJ and its perpendicular bisector. So for the other plane.]

§ 4. **Asymptotes.**

Suppose that in § 3 a curve in the first figure crosses v at Q and t is the tangent at Q, but t and v do not coincide (Fig. 5). Taking t as the line $lx+my+n=0$ referred to in (ii) of § 3 and P as any point of the curve, d/y approaches the limit zero as P approaches Q along the curve.

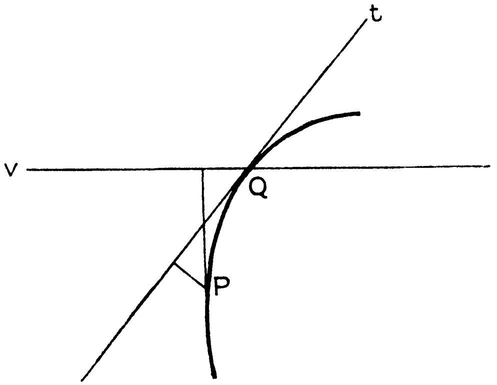

Fig. 5.

Hence the point P' of the corresponding curve in the second figure (the projection of the first), which is at a distance d' from the projection t' of t, approaches t' indefinitely as P' recedes indefinitely.

Such a line as t' which is the projection of a tangent, whose point of contact is on the vanishing line (but the tangent and vanishing line do not coincide), is called an *asymptote* of the projected curve.

§ 5. Circular Points and Lines.

It is well known that the lines through the origin parallel to the asymptotes of a conic are given by equating to zero the terms of the second degree in the Cartesian equation of the conic. Applying this to the general equation of a circle

$$x^2+y^2+2gx+2fy+c=0,$$

we see that the asymptotes of all circles are parallel to

$$x^2+y^2=0;$$

or, as we may put it, all circles go through the unreal and infinitely distant points on $y=\pm\sqrt{(-1)}\,x$.

These points are called the *circular points* at infinity. We shall denote them usually by ω and ω'.

Any line through a circular point, i.e. any line parallel to $y=\pm\sqrt{(-1)}\,x$, is called a *circular line*.

Ex. 1. A circular line is perpendicular to itself.

Ex 2. Two points on a circular line are at zero distance from one another.

Ex. 3. Any point not on a circular line is at an infinite distance from that line.

Ex. 4. Project a given pair of points into the circular points.

[With the line joining them as vanishing line project any conic through them into a circle. See § 2, Ex. 3.]

§ 6 Higher Plane Curves.

A curve whose homogeneous equation is obtained by equating to zero a polynomial in x, y, z of degree n (and whose Cartesian equation is therefore obtained by equating to zero a polynomial in x, y of degree n), is called an *algebraic plane curve* or *higher plane curve* of *degree* n. The word 'higher' implies that the degree is greater than the second, the properties of conics being supposed too well known to require further investigation. Nevertheless we shall not consider conics necessarily excluded from the definition, though most of our work will be concerned with the case $n>2$.

If $f(x, y, z)=0$ is the equation of such a curve, the lines joining its intersections with $\lambda x+\mu y+\nu z=0$ to the point $(0, 0, 1)$ will be $f(\nu x, \nu y, -\lambda x-\mu y)=0$, and are therefore n in number in general. Hence:

*A curve of degree n meets any straight line in n real or unreal points in general.**

---

* We note that the number of real points is $n-2r$, where r is zero or a positive integer; see also § 7.

If the curve meets the line at a point on the vanishing line (§ 2), the projections of the curve and line meet in less than n (finite) points. Such exceptions will be considered later.

A 'curve of degree n' will often be called for short 'an n-ic'. Thus a curve of degree 3, 4, 5, ..., i.e. a cubic, quartic, quintic, ... curve, will be called a 3-ic, 4-ic, 5-ic, ... Another notation for an n-ic in common use is 'C_n'.

Two or more curves whose degrees have the sum n may be considered, when taken together, as forming an n-ic. Such an n-ic will be called *degenerate.* For instance a cubic may degenerate into a conic and a straight line or into three straight lines.

Ex. 1. Through any point O two lines are drawn in fixed directions meeting a given n-ic in $P_1, P_2, \dots, P_n$ and $Q_1, Q_2, \dots, Q_n$. Show that the ratio $OP_1 . OP_2 . \dots . OP_n : OQ_1 . OQ_2 . \dots . OQ_n$ is independent of the position of O.

[Take Cartesian axes of reference in the fixed directions; let O be (x', y'), and the curve be $f(x, y) = 0$. Then

$$OP_1 . OP_2 \dots . OP_n = \pm f(x', y') \div \{\text{coefficient of } x^n \text{ in } f(x, y)\}.]$$

Ex. 2. The sides of a triangle ABC meet an n-ic in $P_1, P_2, \dots, P_n$; $Q_1, Q_2, \dots, Q_n$; $R_1, R_2, \dots, R_n$.

Show that

$$(-1)^n AR_1 . AR_2 . \dots . AR_n . BP_1 . BP_2 . \dots . BP_n . CQ_1 . CQ_2 . \dots . CQ_n$$
$$= AQ_1 . \dots . AQ_n . CP_1 . \dots . CP_n . BR_1 . \dots . BR_n.$$

[Use Ex. 1 or take ABC as triangle of reference. The cases $n = 1$ and 2 are known as Menelaus's and Carnot's theorems.]

Ex. 3. Extend Ex. 2 to the intersections of an n-ic with the sides of any polygon.

Ex. 4. Given all but one of the $3n$ intersections of three lines with an n-ic, construct the remaining intersection.

Ex. 5. The lines joining any point O to two fixed points A and B meet a given n-ic in $P_1, P_2, \dots, P_n$ and $Q_1, Q_2, \dots, Q_n$. Show that the ratio

$$OP_1 . OP_2 . \dots . OP_n . BQ_1 . BQ_2 . \dots . BQ_n :$$
$$OQ_1 . OQ_2 . \dots . OQ_n . AP_1 . AP_2 . \dots . AP_n$$

is independent of the position of O.

[Apply Ex. 2 to the triangle ABO.]

Ex. 6. Two curves touch at P, and O is any point not near P. A line through O close to P meets the curves in Q, R and the tangent at P in T. Show that the ratio of the curvatures of the two curves at P is the cross-ratio of $(OQTR)$.

Deduce the fact that the ratio of the curvatures of two curves at a point of contact is unaltered by projection.

Ex. 7. Given a set of n tangents to a conic meeting at $\frac{1}{2}n(n-1)$ points and a second set of tangents also meeting at $\frac{1}{2}n(n-1)$ points, show that the $n(n-1)$ points thus obtained lie on an $(n-1)$-ic, and that the intersections of an infinite number of n tangents lie on the $(n-1)$-ic.

[Take the conic as the locus of $(t^2, 2t, 1)$. Then, whatever k may be, the intersections of the tangents whose points of contact are given by $f(t)+k\phi(t)=0$, where f and ϕ are polynomials of degree n in t, lie on the $(n-1)$-ic obtained by eliminating t and T from

$$\{f(t).\phi(T)-f(T).\phi(t)\}\div(t-T), \quad x=tTz, \quad y=(t+T)z.$$

The case $n=3$ is well known.]

Ex. 8. If any line meets the curve $x^{2n+1}+y^{2n+1}+z^{2n+1}-xyzu$ in (x_1, y_1, z_1), (x_2, y_2, z_2), .., $(x_{3n+1}, y_{3n+1}, z_{3n+1})$, u being homogeneous in x, y, z of degree $n-3$, then

$$x_1x_2\ldots x_{3n+1}+y_1y_2\ldots y_{3n+1}+z_1z_2\ldots z_{3n+1}=0$$

[If the line is $\lambda x+\mu y+\nu z=0$, then

$$x_1x_2\ldots x_{3n+1}/y_1y_2\ldots y_{3n+1}=(\nu^{2n+1}-\mu^{2n+1})/(\lambda^{2n+1}-\nu^{2n+1}).]$$

Ex 9. A variable line is drawn in a given direction meeting a given n-ic at P, where the radius of curvature is ρ and the tangent makes an angle ϕ with the given direction. Show that $\Sigma\cot\phi$ is constant and $\Sigma(\rho\sin^3\phi)^{-1}=0$, the summation extending over the n intersections P of the line and n-ic.

[For a given value of x we have $\Sigma y=ax+b$, the equation of the n-ic being $y^n-(ax+b)y^{n-1}+\ldots=0$ Differentiate this relation twice with respect to x.]

§ 7. Intersections of Two Curves.

Two curves of degree n and N intersect in nN points.

Suppose the equations of the curves are given in homogeneous coordinates. Let their equations be

$$\left.\begin{aligned} a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\ldots+a_n=0 \\ b_0z^N+b_1z^{N-1}+b_2z^{N-2}+\ldots+b_N=0 \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(i)};$$

where a_r and b_r denote homogeneous polynomials of the rth degree in x and y.

Multiplying these equations respectively by $1, z, z^2, \ldots, z^{N-1}$ and by $1, z, z^2, \ldots, z^{n-1}$, we obtain $n+N$ linear equations in

$$1, z, z^2, \ldots, z^{n+N-1},$$

from which these quantities may be eliminated. For instance, if $n=4$ and $N=3$, we have

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \end{vmatrix}=0. \quad . \quad . \quad \text{(ii)}.$$

This eliminant gives the lines joining the point $(0, 0, 1)$ to the intersections of the two curves, for it is readily seen to be homogeneous in x and y. The lines are nN in number, for a typical term of the determinant just written is $\pm b_0{}^n a_n{}^N$. The theorem is therefore proved.

We note that the number of *real* intersections of the curves is $nN - 2r$, r being zero or a positive integer. For complex roots of (ii) occur in conjugate pairs.

If a_0 and b_0 are zero, both curves pass through $(0, 0, 1)$.

Suppose that $a_0, a_1, \ldots, a_{k-1}$ and $b_0, b_1, \ldots, b_{K-1}$ are identically zero. Multiplying equations (i) by $1, z, z^2, \ldots, z^{N-K-1}$ and by $1, z, z^2, \ldots, z^{n-k-1}$ and eliminating as before, we get an equation giving the lines joining $(0, 0, 1)$ to those intersections of the curves which do not coincide with $(0, 0, 1)$.

For example, if $n = 4$, $N = 3$, a_0 and $a_1 \equiv 0$, $b_0 \equiv 0$, we have

$$\begin{vmatrix} a_2 & a_3 & a_4 & 0 \\ 0 & a_2 & a_3 & a_4 \\ 0 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{vmatrix} = 0.$$

A typical term of the determinant just written is

$$\pm b_K{}^{n-k} a_n{}^{N-K},$$

which is of degree $(n-k)K + (N-K)n = nN - kK$ in x and y.

Hence the curves meet in $nN - kK$ points other than $(0, 0, 1)$. In order to observe the convenient convention that in every case two curves of degrees n and N meet in nN points, we say that kK of the intersections of the curves coincide with $(0, 0, 1)$.

If in this a_k and b_K have a factor in common, the factor is a factor of the first column of the determinant just obtained. It is readily seen that the number of intersections of the curves other than $(0, 0, 1)$ is $nN - kK - 1$ in general, i.e. $kK + 1$ intersections of the curves coincide at $(0, 0, 1)$.

Similarly, if a_k and b_K have r factors in common, $kK + r$ intersections coincide at $(0, 0, 1)$.*

Ex. 1. It is impossible for every line to meet a given n-ic in n real points.

[A line adjacent to and parallel to any tangent will meet it in $n-2$ real points at most.]

Ex. 2. An n-ic with unreal equation cannot pass through more than n^2 real points.

[The only real points on $u + (-1)^{\frac{1}{2}} v = 0$, where $u = 0$ and $v = 0$ are real n-ics, are the real intersections of $u = 0$ and $v = 0$.]

Ex. 3. If a non-degenerate n-ic has the symmetry of a regular polygon of k sides, $n = 2$ or else $n > k$.

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* For a detailed discussion by this method of the intersections of two curves see Segre, *Giornale di Matematiche di Battaglini*, xxxvi (1898), pp. 1–50. For other methods see Ch. VI, § 2, and Ch. IX, § 3. For the general theory of the eliminant (ii) see Böcher, *Higher Algebra*, § 69.

If n is not a multiple of k, the n-ic passes through the circular points.

[Consider the intersections of the n-ic with circles concentric with the polygon.]

Ex. 4. A circle with centre O and radius a meets a given n-ic at $P_1, P_2, \ldots, P_{2n}$. Show that

(i) The sum of the angles which $OP_1, OP_2, \ldots, OP_{2n}$ makes with any fixed line is independent of both a and the position of O.

(ii) The sum of the tangents of the angles which $OP_1, OP_2, \ldots$ make with the tangents at $P_1, P_2, \ldots$ is zero.

(iii) The sum of the polar subtangents of the curve at $P_1, P_2, \ldots$ is zero.

[(i) Use polar coordinates. (ii) and (iii) follow at once from (i). It is assumed that the n-ic does not pass through the circular points.]

Ex. 5. If we eliminate y between two equations $f(x, y) = 0$ and $F(x, y) = 0$ of degrees n and N respectively, we obtain in general an equation for x of degree nN. The coefficient of x^{nN} in this equation only involves the coefficients of the terms of degree n in f and degree N in F. The coefficient of x^{nN-1} only involves the coefficients of the terms of degrees n, $n-1$ in f and degrees N, $N-1$ in F.

Deduce that:

(i) The centroid of $P_1, P_2, \ldots, P_{2n}$ in Ex. 4 is independent of a.

(ii) The centroid of all points on an algebraic curve at which the Cartesian tangent is of length a is independent of a.

(iii) The same is true, if we replace 'Cartesian tangent' by 'Cartesian normal' or 'radius of curvature'.

(iv) The centroid of all points on a curve at which the circle of curvature cuts a circle with centre O and radius a at an angle α is independent of a and α.

§ 8. Pencil of Curves.

If $u = 0$, $v = 0$ are the equations of curves of the nth degree, and k is any constant, $u + kv = 0$ is a curve of degree n passing through the n^2 intersections of $u = 0$ and $v = 0$. The family of curves obtained by taking different values of k is called a *pencil* of n-ics, by analogy with the well-known case in which $n = 1$. The n^2 fixed points through which all curves of the pencil pass are called the *base-points* of the pencil. They are not necessarily all distinct.

Ex. 1. The number of curves of a pencil of n-ics which touch a given line is in general $2(n-1)$.

[Take $z = 0$ as the line.]

Ex. 2. If in Ex. 1 the line passes through a base-point, the number of curves is $2n-3$ in general.

[One touching at the base-point and $2(n-2)$ others.]

§ 9. Tangents.

Suppose $P(x', y', z')$ is any point on a curve $f(x, y, z) = 0$ of degree n, and $Q(x, y, z)$ is any other point. The point dividing PQ in the ratio $\mu : \lambda$, where $\lambda + \mu = 1$, is

$$(\lambda x' + \mu x, \lambda y' + \mu y, \lambda z' + \mu z).$$

This lies on the curve if

$$f(\lambda x' + \mu x, \lambda y' + \mu y, \lambda z' + \mu z) = 0$$

or

$$\lambda^n f(x', y', z') + \lambda^n \mu \left(x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} + z \frac{\partial f}{\partial z'} \right) + \ldots + \mu^n f(x, y, z) = 0 \quad \ldots\ldots\ldots (\text{i});$$

where $\frac{\partial f}{\partial x'}$ means the result of putting x' for x, y' for y, z' for z in $\frac{\partial f(x, y, z)}{\partial x}$, &c.

This equation in μ/λ gives the ratio in which the curve divides PQ. One root is zero, for P lies on the curve. A second root is zero, if PQ touches the curve at P. The condition for this is

$$x \frac{\partial f}{\partial x'} + y \frac{\partial f}{\partial y'} + z \frac{\partial f}{\partial z'} = 0,$$

which is therefore the equation of the tangent to the curve at P.

If the equation of the curve in Cartesian coordinates is $f(x, y) = 0$, the tangent at (x', y') is

$$(x - x') \frac{\partial f}{\partial x'} + (y - y') \frac{\partial f}{\partial y'} = 0.$$

If the coordinates of any point of a curve are given in the form $\quad x = f(t), \; y = \phi(t), \; z = \psi(t),$

where t is called the *parameter* of the point, the tangent at the point is

$$(\phi\psi' - \phi'\psi) x + (\psi f' - \psi' f) y + (f\phi' - f'\phi) z = 0.$$

In fact, this line goes through the given point and the consecutive point, whose coordinates are

$(f(t + dt), \phi(t + dt), \psi(t + dt))$ or $(f + f'dt, \phi + \phi'dt, \psi + \psi'dt)$, neglecting the squares, cubes, &c., of dt.

If we put $z = 1$, we get the tangent at any point of the curve whose Cartesian coordinates are

$$x = f/\psi, \; y = \phi/\psi.$$

Suppose now $P(x', y', z')$ is not on the curve. The line PQ

will touch the curve, if the equation (i) in μ/λ has equal roots. If we write down the condition for this, the resulting equation is that of the tangents from (x', y', z') to the curve.

Ex. 1. If the n-ic $f(x, y, z) = 0$ passes through $(0, 0, 1)$, the coefficient of z^n in f is zero, and the tangent at $(0, 0, 1)$ is obtained by equating to zero the terms involving z^{n-1}.

If $f(x, y) = 0$ passes through the origin, the constant term in f is zero and the tangent at the origin is obtained by equating to zero the terms of the first degree in f

Ex. 2. The axes of reference intercept a constant length on any tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 3. The tangent at any point P of $yx^2 = a^3$ meets the axes in A, B and the curve again at Q. Prove that the ratios $AP : PB : BQ$ are constant.

Ex. 4. The sum of the intercepts made by any tangent to $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ on the axes of reference is constant.

Ex. 5. Show that two perpendicular tangents to

$$x = a\,(2\cos t + \cos 2t), \qquad y = a\,(2\sin t - \sin 2t)$$

meet on the circle $x^2 + y^2 = a^2$.

Ex. 6. The tangents from any point O to a cubic are six in number in general. Their points of contact lie on a conic, and they meet the cubic again in six points on a conic. The tangents meet the two conics again in twelve points on another cubic also touching the tangents.

[Putting (i) in the form $(n = 3)$, $u_0\lambda^3 + 3u_1\lambda^2\mu + 3u_2\lambda\mu^2 + u_3\mu^3 = 0$, where u_r is of degree r in x, y, z, and writing down the condition for equal roots, we have

$$u_3\,(u_0{}^2u_3 - 6\,u_0u_1u_2 + 4\,u_1{}^3) + u_2{}^2\,(4\,u_0u_2 - 3\,u_1{}^2) = 0$$

The two conics are $u_2 = 0$ and $4u_0u_2 = 3u_1{}^2$; and the other cubic is

$$u_0{}^2u_3 - 6\,u_0u_1u_2 + 4\,u_1{}^3 = 0\,]$$

Ex. 7. The two cubics of Ex. 6 are in plane harmonic perspective with O as vertex, each conic corresponding to itself in the perspective. (See § 3, Ex. 1.)

[Choose the triangle of reference so that O is $(0, 0, 1)$ and $u_2 \equiv z^2 + 2xy$.]

Ex. 8. Find the lengths of the tangents and normals from any point to a curve.

[Take the point as origin, and put the equation of the curve into the form $f(r, t) = 0$, where $x = (1 - t^2)\,r/(1 + t^2)$, $y = 2tr/(1 + t^2)$. Eliminate t from $f = 0$, and $\dfrac{\partial f}{\partial r} = 0$, or $\dfrac{\partial f}{\partial t} = 0$.]

§ 10. Inversion.

If O, P, P' are collinear points, O being fixed, such that $OP \,.\, OP' = k^2$, the locus of P and the locus of P' are said to be *inverses* of each other with respect to a circle* with centre O and radius k: or more simply 'inverses with respect to O'.

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* Or a sphere, if the loci are not coplanar.

If P and P', Q and Q' are points on two inverse curves, $OP . OP' = OQ . OQ'$. Hence the angles OPQ, $OQ'P'$ are equal. Making Q consecutive to P and therefore Q' consecutive to P', we see that the tangents at P and P' to the inverse curves are coplanar and make supplementary angles with OPP'.

Again, from the similar triangles OPQ, $OQ'P'$ (Q not necessarily consecutive to P) we have

$$PQ : P'Q' = OP : OQ' = OP.OP' : OP'.OQ'.$$

Hence $$PQ = k^2 . P'Q'/OP'.OQ',$$

which enables us to express lengths in one figure in terms of lengths in the inverse figure.

Suppose now that two curves meet in P and the two inverse curves in P'.

Let the tangents at P and P' to one pair of inverse curves meet in H and the tangents to the other pair in K. Then, since the tangents HP, HP' make supplementary angles with OPP', $PH = P'H$ and similarly $PK = P'K$. Therefore the triangles HPK, $HP'K$ are congruent, and the angles HPK, $HP'K$ are equal.

Hence two curves cut at the same angle as their inverses.

If now the two inverse figures are coplanar while O is taken as the origin of rectangular Cartesian axes and P is the point (x, y), P' is $(k^2 x/(x^2 + y^2), k^2 y/(x^2 + y^2))$. This follows at once from the fact that O, P, P' are collinear and $OP . OP' = k^2$.

Hence, if $f(x, y) = 0$ is the equation of a plane curve,

$$f(k^2 x/(x^2 + y^2), k^2 y/(x^2 + y^2)) = 0$$

is the equation of the inverse curve.

For instance the inverse of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

is $$k^4 + 2k^2 (gx + fy) + c(x^2 + y^2) = 0.$$

Hence the inverse of a circle with respect to a point O in its plane is a circle, or a straight line if O lies on the given circle.

The inverse of a straight line is in general a circle through O. But, if the line passes through O, it is evidently its own inverse. Also, if the line passes through a circular point, its inverse is a line through the other circular point. In fact, the inverse of $$x + iy = c$$

is $$k^2 = c(x - iy).$$

We have a similar result for three dimensions, and may show that the inverse of a sphere with respect to O is a sphere, or a plane if O lies on the given sphere.

The inverse of a circle with respect to a point not in its plane is a circle, for the intersection of two spheres inverts into the intersection of two spheres.

Ex. 1. Prove that a circle and two inverse points invert into a circle and two inverse points.

[Use the fact that all circles through two points inverse with respect to a circle cut this circle orthogonally.]

Ex. 2. A sphere is projected from any point V on its surface on to the diametral plane perpendicular to the radius through V. Show that circles project into circles and angles are unaltered by the projection.

[The projection is the inverse of the original figure with respect to V. This projection is called *stereographic*.]

Ex. 3. If ω, ω' are the circular points, the inverse with respect to O of a point P on $O\omega$ is at ω.

If a curve cuts $O\omega$ at P, the corresponding tangent at ω of the inverse curve is the inverse of $\omega' P$.

[Let Q be a point of the curve near P and Q' its inverse. Then $\omega Q'$ and $\omega' Q$ are inverse lines. Now let Q approach P.]

§ 11. Theory of Equations.

For convenience of reference we insert here some well-known results in the theory of equations.

The typical equation of degree n is

$$a_0 x^n + {}^nC_1 a_1 x^{n-1} + {}^nC_2 a_2 x^{n-2} + \ldots + a_n = 0.$$

The product of two roots is -1 if

$$a_0 + a_2 = 0, \quad a_0(a_0 + 3a_2) + a_3(3a_1 + a_3) = 0,$$

$$(a_0 + 6a_2 + a_4)(a_0 - a_4)^2 + 16(a_1 + a_3)(a_0 a_2 + a_1 a_4) = 0,$$

$$\begin{aligned}\{a_0(a_0 - 5a_4) + a_5(a_5 - 5a_1)\}\{a_0 + 10a_2 + 5a_4)(a_0 - 5a_4) \\ + (a_5 + 10a_3 + 5a_1)(a_5 - 5a_1)\} \\ + 25(a_0 a_1 + 2a_0 a_3 - a_5 a_4 - 2a_5 a_2)^2 = 0, \ \ldots\end{aligned}$$

in the cases $n = 2, 3, 4, 5, \ldots$.

The cubic equation ($n = 3$) has two invariants

$$G \equiv a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3, \quad H \equiv a_0 a_2 - a_1^2.$$

According as

$$G^2 + 4H^3 \equiv a_0^2 (a_0^2 a_3^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3 - 3a_1^2 a_2^2)$$

is negative, positive, or zero, the cubic has three real roots, one real root, or two equal roots.

The quartic equation ($n = 4$) has two invariants

$$I \equiv a_0 a_4 - 4a_1 a_3 + 3a_2^2, \qquad J \equiv \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

If ϕ is the cross-ratio $(\alpha-\beta)(\gamma-\delta) \div (\alpha-\delta)(\gamma-\beta)$ of the roots α, β, γ, δ of the quartic,

$$\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2 I^3 = 27(\phi^2-\phi+1)^3 J^2.$$

We note the cases $\phi = 0, -1, \frac{1}{2}(1 \pm \sqrt{-3})$.

If $\phi = 0$, the quartic has two equal roots and $I^3 = 27J^2$.

If $\phi = -1$, $J = 0$; and if $\phi = \frac{1}{2}(1 \pm \sqrt{-3})$, $I = 0$.*

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* The range of points $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$, $(\delta, 0)$ and the pencil of lines $y = \alpha x$, $y = \beta x$, $y = \gamma x$, $y = \delta x$ are said to be respectively 'harmonic' or 'equianharmonic' in these two cases.

CHAPTER II

SINGULAR POINTS

§ 1. Inflexions, &c.

Suppose that the Cartesian equation of a curve of degree n arranged in ascending powers of x and y is

$$0 = a + b_0x + b_1y + c_0x^2 + 2c_1xy + c_2y^2 + d_0x^3 + \dots .$$

If the origin lies on the curve, the equation must be satisfied by $x = 0$ and $y = 0$; and therefore $a = 0$.

The equation now becomes

$$0 = b_0x + b_1y + c_0x^2 + 2c_1xy + c_2y^2 + d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3 + e_0x^4 + \dots .$$

The line $y = mx$ meets the curve where

$$0 = x(b_0 + b_1m) + x^2(c_0 + 2c_1m + c_2m^2) + x^3(d_0 + 3d_1m + 3d_2m^2 + d_3m^3) + x^4(e_0 + \dots) + \dots .$$

One root of this equation in x is always zero; as was to be expected, since any line through the origin meets the curve in one point coinciding with the origin (Fig. 1).

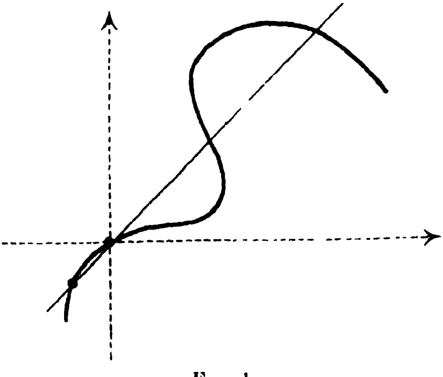

Fig. 1.

Suppose that, as m is made to vary, $b_0 + b_1m$ approaches zero. Then a second root of the equation approaches zero, i.e. a second intersection of $y = mx$ with the curve approaches the origin, or the line becomes the tangent at the origin (Fig. 2).

Eliminating m between $y = mx$ and $b_0 + b_1 m = 0$, we obtain the equation of the tangent at the origin, namely,

$$b_0 x + b_1 y = 0.$$

Suppose that, when $b_0 + b_1 m = 0$, we have also

$$c_0 + 2c_1 m + c_2 m^2 = 0.$$

Then the tangent at the origin meets the curve at three points coinciding with the origin (Figs. 3 and 4). The origin is called a *point of inflexion* (or simply *an inflexion*) in this case. The tangent at the origin is called an *inflexional tangent* or *tangent of three-point contact*.

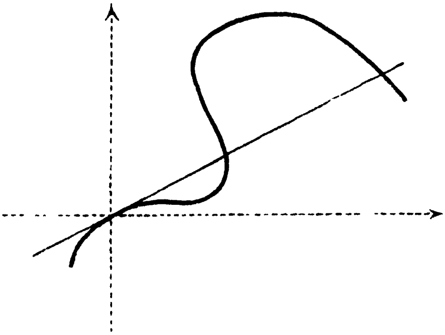

Fig. 2.

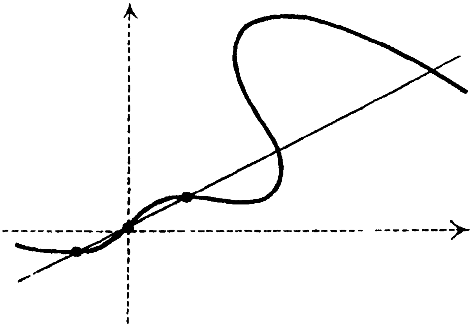

Fig. 3.

It is evident that $b_0 + b_1 m$ is a factor of $c_0 + 2c_1 m + c_2 m^2$; or, what is the same thing,

$$b_0 x + b_1 y \text{ is a factor of } c_0 x^2 + 2c_1 xy + c_2 y^2.$$

If we have also

$$d_0 + 3d_1 m + 3d_2 m^2 + d_3 m^3 = 0,$$

the tangent meets the curve in four points coinciding with the origin. The origin is sometimes called a *point of undulation* in this case. The tangent is called a *tangent of four-point contact*.

In this case b_0x+b_1y is a factor of both

$c_0x^2+2c_1xy+c_2y^2$ and of $d_0x^3+3d_1x^2y+3d_2xy^2+d_3y^3$.

These results can be immediately generalized and we have:

If a curve passes through the origin, the terms of the first degree equated to zero give the tangent at the origin.

If the terms of the first degree are a factor of the terms of the 2^{nd}, 3^{rd}, ..., $(r-1)^{\text{th}}$ *degrees, the tangent meets the curve in* r *points coinciding with the origin, and is called a 'tangent of* r*-point contact'.*

The curve crosses the tangent at the point of contact if r *is odd, but does not cross it if* r *is even.*

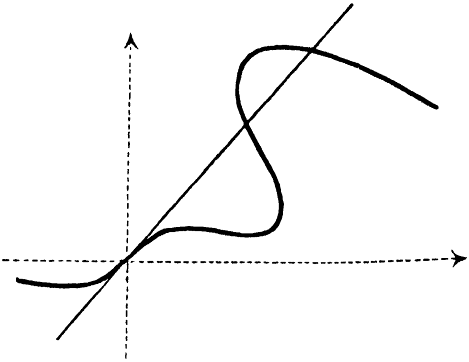

Fig. 4.

The last statement in the theorem may be proved as follows:

The curve may be written

$$b_0x+b_1y=(u_r+u_{r+1}+\dots+u_n)\div(1+v_1+v_2+\dots+v_{r-2}),$$

where u_k and v_k are homogeneous of degree k in x and y.

The perpendicular from any point (x, y) of the curve on the tangent at the origin is

$$(b_0x+b_1y)\div(b_0^2+b_1^2)^{\frac{1}{2}}$$
$$=(u_r+u_{r+1}+\dots+u_n)\div(1+v_1+\dots+v_{r-2})(b_0^2+b_1^2)^{\frac{1}{2}},$$

and has therefore the same sign as $u_r\div(b_0^2+b_1^2)^{\frac{1}{2}}$, when x and y are small and approximately in the ratio $b_1:-b_0$. But u_r evidently does or does not change sign with x and y, where $b_0x+b_1y=0$, according as r is odd or even.

The truth of the statement may also be seen geometrically by considering a tangent of r-point contact as the limiting position of a line meeting the curve in r points close together (see Fig. 3 for the case $r = 3$).

The definition of 'r-point contact' of a line and a curve may be generalized. Any two curves will have 'r-point contact' at P, if they may be considered as the limiting case of two curves meeting one another at r points * very close to P. Curves having three-point contact at P are said to *osculate* at P. For instance, at each point of a curve there is an osculating circle called the 'circle of curvature', whose centre and radius are the 'centre of curvature' and 'radius of curvature' of the given curve at P. The circle of curvature has four-point contact with the curve, if the radius of curvature of the curve is a maximum or minimum at P.†

§ 2. Double Points.

Suppose that in § 1 b_0 and b_1 are zero. The curve is now

$$0 = c_0x^2 + 2c_1xy + c_2y^2 + d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3 + e_0x^4 + \dots;$$

and $y = mx$ meets the curve where

$$0 = x^2(c_0 + 2c_1m + c_2m^2) + x^3(d_0 + 3d_1m + 3d_2m^2 + d_3m^3) + x^4(e_0 + \dots) + \dots.$$

Two roots of this equation are zero for every value of m.

Hence every line through the origin meets the curve in two points coinciding with the origin. The origin is called a *double point.*

The tangents at the origin are defined as the lines meeting the curve at three points coinciding with the origin. The line $y = mx$ is such a tangent if

$$c_0 + 2c_1m + c_2m^2 = 0.$$

Eliminating m between this equation and $y = mx$, we have the equation of the tangents at the origin

$$c_0x^2 + 2c_1xy + c_2y^2 = 0.$$

Such a tangent may be considered as the limiting position of a chord joining the double point to another point of the curve, when this point approaches the double point.

There are three cases to be discussed.

The tangents may be real, when the origin is called a *crunode* (Fig. 5). In this case the curve has two real 'branches' through the origin.

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* On the same branch of each curve; see next section.

† See treatises on 'Differential Calculus'.

The tangents may be unreal, when the origin is called an *acnode* (Fig. 6). The origin is a real point of the curve, but there is no real point of the curve adjacent to the origin.

The tangents may be coincident, when the origin is called a *cusp* (Fig. 7).

The word *node* is equivalent to 'crunode or acnode', i. e. to 'a double point which is not a cusp'.*

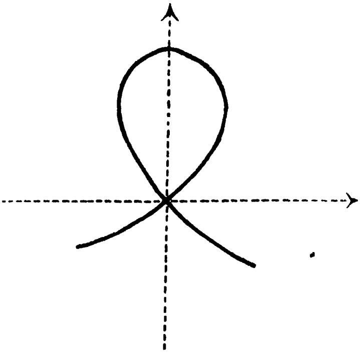

Fig. 5.

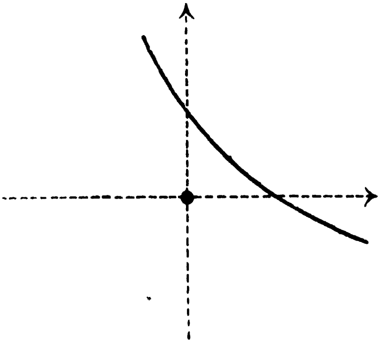

Fig. 6

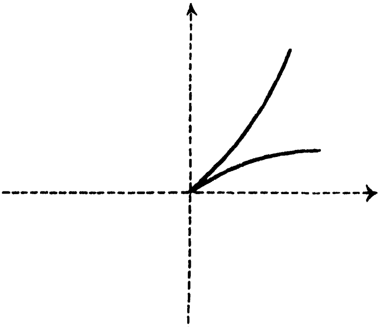

Fig. 7.

As in § 1, we show that, if one factor of $c_0x^2+2c_1xy+c_2y^2$ is a factor of $d_0x^3+3d_1x^2y+3d_2xy^2+d_3y^3$, the corresponding tangent is an inflexional tangent. The origin is called a *flecnode* in this case (Fig. 8).

If both tangents are inflexional tangents, so that the terms of the second degree in the equation of the curve are a factor of the terms of the third degree, the origin is called a *biflecnode* (Fig. 9).

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* Other nomenclatures are 'node' for crunode, 'isolated point' or 'conjugate point' for acnode. The term 'spinode' for cusp is obsolete.

Similarly if one of the factors (or both) of the terms of the second degree is a factor of the terms of the third, fourth, . . . degrees.

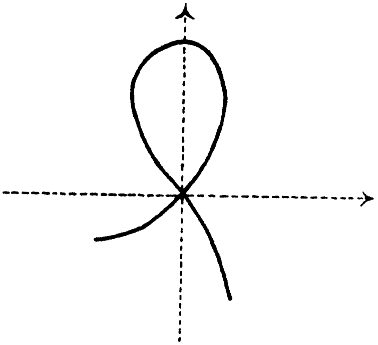

Fig. 8.

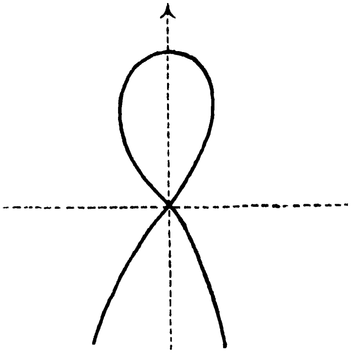

Fig. 9.

§ 3. Multiple Points.

If c_0, c_1, c_2 are all zero, so that the terms of the lowest degree are the terms of the third degree, the origin is called a *triple point*. Any line through the origin meets the curve in three points coinciding with the origin, except the three tangents at the origin obtained by equating to zero the terms of lowest degree. These tangents meet the curve at four points (at least) coinciding with the origin.

In general we have the result:

If the terms of lowest degree in the Cartesian equation of a curve are of degree k, the origin is called a multiple point of order k (a k-ple point). Any line through the origin meets the curve in k points coinciding with the origin, except the k tangents at the origin obtained by equating to zero the terms of lowest degree, which meet the curve in $k+1$ points (at least) coinciding with the origin.

If the equation of the curve is given in homogeneous coordinates, a very slight modification is necessary. Let us suppose the curve is of degree n, and passes through the point (0, 0, 1). Exactly as in §§ 1 and 2 the consideration of the intersections of the curve with the line $y = mx$ shows that:

If the highest power of z which occurs in the homogeneous equation of a curve of degree n is z^{n-k}, the point (0, 0, 1) *is a multiple point of order k, and the tangents at* (0, 0, 1) *are* *obtained by equating to zero the terms multiplying* z^{n-k} *in the equation. If the terms multiplying* $z^{n-k}, z^{n-k-1}, \ldots, z^{n-k-r}$ *have a factor in common, the corresponding tangent has* $(r+2)$*-point contact with the branch it touches.*

Ex. 1. Investigate the nature of the origin and trace roughly the curves

$$a^2y = x^3, \quad ay^2 = x^3, \quad a^3y = x^4, \quad ay^3 = x^4, \quad y^2(a-x) = x^3,$$
$$(x^2+y^2)^2 = a^2(x^2-y^2), \quad (x^2+y^2)^2 = ay(3x^2-y^2), \quad x^4+y^4 = axy^2,$$
$$a^4y^4 + x^8 = 2a^3x^3y^2.$$

[Inflexion, cusp, undulation, triple point, cusp, biflecnode (turn into polars to trace), triple point (turn into polars to trace), triple point (solve for y to trace), quadruple point (solve for y to trace).]

Ex. 2. Trace roughly the curve $y^2 + x^3(x^2-1)^2(x-2) = 0$.

[Acnode at $(-1, 0)$, cusp at $(0, 0)$, crunode at $(1, 0)$. Transfer the origin to each point in turn.]

Ex. 3. Trace $y^2 = x^2(x+k)$ when $k = -1, 0, 1$.

Ex. 4. Trace $y^2 + x^2(x^2 - 2x + k) = 0$ when $k = -3, 0, \frac{1}{2}, 1$.

Ex. 5. Trace $y^2 + x(x-k)(x-2)^2(x-4)^2 = 0$ when $k = -1, 0, 1, 2, 3, 4, 5$.

Ex. 6. An n-ic has the sides CA, CB of the triangle of reference as tangents of r-point contact, A and B being the points of contact. Show that its equation is of the form $xyu_{n-2} = z^r u_{n-r}$, where u_t is homogeneous of degree t in x, y, z.

[The equation has z^r as a factor when we put $x = 0$ or $y = 0$. Ex. 7 to 10 are special cases.]

Ex. 7. The line joining two real inflexions of a cubic passes through a third real inflexion.

[In Ex. 6 $n = 3$, $r = 3$.]

Ex. 8. The line joining two inflexions A, B of a quartic meets the curve again in D, E. The tangents at A, B meet the curve again in P, Q, and PQ meets the curve again at R, S. Show that a conic can be drawn osculating the quartic at D, E and passing through R, S.

[$n = 4$, $r = 3$.]

Ex. 9. The line joining two undulations of a 4-ic meets the curve again in P, Q. Show that a conic can be drawn having four-point contact with the curve at P and Q.

[$n = 4$, $r = 4$.]

Ex. 10. The line through three real collinear undulations of a quartic passes through a fourth real undulation.

Ex. 11. Three tangents of n-point contact of an n-ic are taken as the sides of the triangle of reference. Show that the equation of the n-ic is of the form

$$xyzu_{n-3} + (ax \pm by)^n + (by+cz)^n + (cz+ax)^n = a^nx^n + b^ny^n + c^nz^n;$$

the + sign in the ambiguity being taken if n is odd, and either sign if n is even.

Ex. 12. An n-ic has three tangents having n-point contact. If n is odd, the three points of contact are collinear. If n is even, either the points are collinear or the three lines joining each to the intersection of the tangents at the other two are concurrent. If the tangents are concurrent, only the former alternative is possible.

Discuss the cases $n = 2, 3, 4$.

[Use Ex. 11.]

Ex. 13. An n-ic has r-point contact with the side BC of the triangle of reference ABC at B and C. Show that its equation is of the form

$$zu_{n-1} = x^r y^r u_{n-2r}.$$

Ex. 14. A line is drawn through each of the points of contact of a *bitangent* of a quartic (a line touching the quartic at two points; see Ch. IV, § 7). Show that a cubic touches the quartic at the six points in which these two lines meet the curve again.

[In Ex. 13 $n = 4$, $r = 2$.]

Ex. 15. Enunciate a similar theorem for a sextic having three-point contact with a line at two points.

Ex. 16. An n-ic has r-point contact with each of $x = 0$ and $y = 0$ at two distinct points. Show that its equation is of the form

$$xyu_{n-2} = u_2{}^r u_{n-2r}.$$

[See Ch. XIX, § 2.]

Ex. 17. If the six points of contact of three bitangents of a 5-ic, 6-ic, or 7-ic lie on a conic, the other intersections of the bitangents with the curve are respectively collinear, lying on a conic, the base points of a pencil of 3-ics.

[Use Ch. I, § 6, Ex. 2 in Ex. 17, 18, 19.]

Ex. 18. The tangents at three collinear inflexions of a quartic meet the curve again in collinear points.

What are the corresponding theorems for a 5-ic and 6-ic?

Ex. 19. Three bitangents of a quartic form a triangle ABC whose sides touch the quartic at P_1 and P_2, Q_1 and Q_2, R_1 and R_2. Show that the conic $P_1Q_1Q_2R_1R_2$ passes either through P_2 or through the harmonic conjugate of P_2 with respect to B and C.

What are the corresponding theorems for a $2n$-ic having n-point contact with each of three lines at two points; for a 6-ic with three triple tangents, &c.?

Ex. 20. The tangents to an n-ic from the k-ple point $(0, 0, 1)$ are found by writing down the condition that the equation of the curve, considered as an equation in z, should have equal roots. There are $n(n-1)-k(k+1)$ such tangents in general.

§ 4. Conditions for a Double Point.

Let (X, Y) be any point on the curve whose Cartesian equation is $f(x, y) = 0$. Transfer the origin to this point, and the equation becomes $f(x+X, y+Y) = 0$, or

$$0 = f(X, Y) + x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y}$$
$$+ \tfrac{1}{2}\left(x^2\frac{\partial^2 f}{\partial X^2} + 2xy\frac{\partial^2 f}{\partial X\,\partial Y} + y^2\frac{\partial^2 f}{\partial Y^2}\right) + \dots,$$

where $\frac{\partial f}{\partial X}$ means the result of putting X for x and Y for y in $\frac{\partial f}{\partial x}$, &c. If the new origin is a double point on the curve,

$$f(X, Y) = \frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = 0,$$

and the tangents at the new origin are

$$x^2 \frac{\partial^2 f}{\partial X^2} + 2xy \frac{\partial^2 f}{\partial X \partial Y} + y^2 \frac{\partial^2 f}{\partial Y^2} = 0.$$

Hence, with a slight change of notation,

If (x, y) is a double point on the curve $f(x, y) = 0$,

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

The double point is a crunode, acnode, or cusp, according as

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 >, <, \text{ or } = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}.$$

Similarly, if (x, y) is a k-ple point, all the partial derivatives of f with respect to x and y up to the order $k-1$ inclusive must vanish.

Again, let (X, Y, Z) be any point O on the curve of degree n whose homogeneous equation is $f(x, y, z) = 0$. Let $P(x, y, z)$ be any other point, and let Q be the point

$$\left(\frac{\lambda x + \mu X}{\lambda + \mu}, \frac{\lambda y + \mu Y}{\lambda + \mu}, \frac{\lambda z + \mu Z}{\lambda + \mu}\right)$$

dividing PO in the ratio $\mu : \lambda$. This point lies on the curve if

$$f(\lambda x + \mu X, \lambda y + \mu Y, \lambda z + \mu Z) = 0,$$

i. e.

$$\mu^n f(X, Y, Z) + \mu^{n-1}\lambda\left(x \frac{\partial f}{\partial X} + y \frac{\partial f}{\partial Y} + z \frac{\partial f}{\partial Z}\right)$$
$$+ \tfrac{1}{2}\mu^{n-2}\lambda^2\left(x^2 \frac{\partial^2 f}{\partial X^2} + y^2 \frac{\partial^2 f}{\partial Y^2} + z^2 \frac{\partial^2 f}{\partial Z^2} + 2yz \frac{\partial^2 f}{\partial Y \partial Z}\right.$$
$$\left. + 2zx \frac{\partial^2 f}{\partial Z \partial X} + 2xy \frac{\partial^2 f}{\partial X \partial Y}\right) + \ldots + \lambda^n f(x, y, z) = 0.$$

If O is a double point, two roots of this equation in λ/μ are zero for all values of $x : y : z$; or

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0.$$

It will be noticed that

$$\frac{\partial f}{\partial X} = \frac{\partial f}{\partial Y} = \frac{\partial f}{\partial Z} = 0$$

implies $f(X, Y, Z) = 0$, since by Euler's theorem on homogeneous functions

$$X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z} \equiv n \, . f(X, Y, Z).$$

Three roots of the equation in λ/μ are zero if P lies on a tangent at O. Therefore the equation of the tangents at O is

$$x^2\frac{\partial^2 f}{\partial X^2} + y^2\frac{\partial^2 f}{\partial Y^2} + z^2\frac{\partial^2 f}{\partial Z^2} + 2yz\frac{\partial^2 f}{\partial Y \partial Z} + 2zx\frac{\partial^2 f}{\partial Z \partial X} + 2xy\frac{\partial^2 f}{\partial X \partial Y} = 0.$$

Hence:

If (x, y, z) is a double point on the curve $f(x, y, z) = 0$,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

The double point is a crunode, acnode, or cusp, according as the lines

$$\xi^2\frac{\partial^2 f}{\partial x^2} + \eta^2\frac{\partial^2 f}{\partial y^2} + \zeta^2\frac{\partial^2 f}{\partial z^2} + 2\eta\zeta\frac{\partial^2 f}{\partial y \partial z} + 2\zeta\xi\frac{\partial^2 f}{\partial z \partial x} + 2\xi\eta\frac{\partial^2 f}{\partial x \partial y} = 0$$

are real, unreal, or coincident; ξ, η, ζ being current coordinates.

The reality, unreality, or coincidence of the lines is at once determined by considering their intersections with $\xi = 0$, $\eta = 0$, or $\zeta = 0$.

We may deduce the result for Cartesian coordinates by putting $z = 1$; or the result for homogeneous coordinates may be deduced from that for Cartesian coordinates by replacing x and y by x/z and y/z.

Ex. 1. Find the multiple points of $f \equiv y^3(4a-y)^3 - 4x^4(x+3a)^2 = 0$.

[Equating to zero $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial a}$ we obtain $(x+2a)(x+3a)x^3 = 0$, $(y-2a)(4a-y)^2y^2 = 0$, $y^3(4a-y)^2 = 2x^4(x+3a)$. These are all satisfied at the points $(0, 0)$, $(0, 4a)$, $(-3a, 0)$, $(-3a, 4a)$, $(-2a, 2a)$. Transferring the origin to each of these points in turn, we see that the first two are triple points, the next two are cusps, and the last is a crunode.]

Ex. 2. Find the double points of

(i) $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0$.

(ii) $x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0$.

(iii) $x^2y^2 + 12a^3(3x+2y) + 108a^4 = 0$.

(iv) $4(x-1)^3+(y-3x+2)^2=0$.

(v) $(x^2+y^2-a^2)^3+27x^2y^2a^2=0$.

(vi) $(3axy+2x^3+c^3)^2=4(ay+x^2)^3$.

(vii) $(x^2-a^2)^2y=(y^2-b^2)^2x$.

(viii) $(x^2-a^2)^2+(y^2-b^2)^2=a^4$.

(ix) $(8y-x-a)^3=216axy$.

(x) $x^2y^2-4a(x^3+y^3)+18a^2xy-27a^4=0$.

(xi) $y(x+3)^2=4(4x-3y)(2x-3y-6)$.

[(i) Crunodes at $(a, 0)$, $(-a, 0)$, $(0, -a)$.

(ii) Crunodes at $(0, a)$, $(a, 0)$, $(2a, a)$.

(iii) Crunodes at $(-2a, -3a)$, cusps at $(\infty, 0)$, $(0, \infty)$. For these latter replace a by z. Then the tangents at $(1, 0, 0)$ are the coefficient of x^2 equated to zero, i.e. $y^2=0$. Or we may project the curve as in Ch. I, § 3. See Ch. X, Fig. 1.

(iv) Cusp at $(1, 1)$.

(v) Cusps at $(\pm a, 0)$, $(0, \pm a)$ and the circular points. Unreal nodes along $x=\pm y$. The curve is the 'four-cusped hypocycloid'.

(vi) Cusp at $(c, -c^2/a)$.

(vii) Nodes at $(\pm a, \pm b)$.

(viii) Nodes at $(0, \pm b)$.

(ix) Node at $(a, -a/8)$.

(x) Cusps at $(3a, 3a)$, $(3a\omega^2, 3a\omega)$, $(3a\omega, 3a\omega^2)$.

(xi) Cusp at $(-3, -4)$.]

Ex. 3. Find the double points of

(i) $a/x^2+b/y^2+c/z^2+2f/yz+2g/zx+2h/xy=0$.

(ii) $\Sigma yz(y^2z+yz^2-2x^3+24xyz)=0$.

(iii) $\Sigma yz(y^2z+yz^2-4x^3+2xyz)=0$.

[(i) Nodes at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.

(ii) Cusps at $(1, 0, 0)$, &c.; nodes at $(-1, 5, 5)$, &c.

(iii) Triple point at $(1, 1, 1)$; nodes at $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.]

Ex. 4. When has $f \equiv y^2z-4x^3+g_2xz^2+g_3z^3=0$ a double point?

[$\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=\dfrac{\partial f}{\partial z}=0$ gives $yz=12x^2-g_2z^2=y^2+2g_2xz+3g_3z^2=0$. These are only satisfied if $y=0$, $2g_2x+3g_3z=0$, $g_2^3=27g_3^2$. Hence the required condition is $g_2^3=27g_3^2$. If this holds, there is a double point at $(-\frac{1}{2}g_3^{\frac{1}{3}}, 0, 1)$, which is a crunode or acnode as g_3 is <0 or >0.]

Ex. 5. For what value of k have the following curves a double point?

(i) $z^2x=y(y-x)(y-kx)$.

(ii) $x^3+y^3+z^3=k(x+y+z)^3$.

(iii) $(x+y+z)^3+6kxyz=0$.

(iv) $x^3+y^3+z^3+6kxyz=0$.

(v) $x^4+xy^3+y^4-kzy^3-2x^2yz-xy^2z+y^2z^2=0$.

[(i) $k=0$; an acnode at $(1, 0, 0)$. $k=1$; a crunode at $(1, 1, 0)$.

(ii) $k=\frac{1}{9}$; an acnode at $(1, 1, 1)$.

(iii) $k = -\frac{3}{2}$; an acnode at $(1, 1, 1)$.

(iv) $8k^3 = -1$. The cubic splits up into three lines.

(v) Always a cusp at $(0, 0, 1)$. A node in addition if $k = 1$ or 2.]

Ex. 6. For what values of a, b, c has $z^3 = (ax+by+cz)xy$ a double point?

[$a = 0$, $(1, 0, 0)$; $b = 0$, $(0, 1, 0)$; $c^3 = 27ab$, $(bc, ca, -3ab)$.]

Ex. 7. Of a pencil of n-ics $3(n-1)^2$ have a node in general.

If every curve of a pencil has a double point at O, the tangents at O form an involution.

Ex. 8. If $S_1 = 0$, $S_2 = 0$, ..., $S_r = 0$ have a k-ple point at P and $C_1 = 0$, $C_2 = 0$, . ., $C_r = 0$ are any other curves,

$$f \equiv C_1 S_1 + C_2 S_2 + \dots + C_r S_r = 0$$

has a k-ple point at P.

[$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ is satisfied at P.]

Ex. 9. If $u = 0$, $v = 0$ are lines and $p = 0$, $q = 0$, $r = 0$ are $(n-2)$-ics, $pu^2 + 2quv + rv^2 = 0$ is an n-ic with a node at the intersection of $u = 0$ and $v = 0$.

[See Ex. 8.]

Ex. 10. In symmetrical form the conditions that (x, y, z) should be a node of $f = 0$ are $f_1 = f_2 = f_3 = 0$, and that it should be a cusp are

$$f = f_{11}.f_{23} - f_{12}.f_{13} = f_{22}.f_{31} - f_{23}.f_{21} = f_{33}f_{12} - f_{31}.f_{32} = 0;$$

where the suffixes 1, 2, 3 denote partial differentiation with respect to x, y, z.

[For the cusp, we get $f_{22}.f_{33} = f_{23}^2$, &c. Then

$$(n-1)(f_{12}.f_1 - f_{11}f_2) = f_{12}(xf_{11} + yf_{12} + zf_{13}) - f_{11}(xf_{21} + yf_{22} + zf_{23}) = 0.$$

Hence $f_1/f_{11} = f_2/f_{12} = f_3/f_{13} = nf/(n-1)f_1$; which gives

$$f_1 = f_2 = f_3 = 0.$$

We leave to the reader the modifications required when some of the quantities f_{11}, f_{23}, ... are zero.]

§ 5. Points at Infinity.

Suppose that in the equation

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_n = 0$$

the coefficients are made to vary. Then, if p_0 approaches the limit zero, one of the roots of the equation becomes indefinitely large.

For the equation is

$$p_0 + p_1\xi + p_2\xi^2 + \dots + p_n\xi^n = 0,$$

where $x\xi = 1$.

Now if $p_0 \to 0$, one value of ξ given by the latter equation tends to zero. Hence one value of x given by the former equation becomes very large.

Similarly, if $p_0, p_1, \dots, p_r$ all $\to 0$, $r+1$ roots of the equation in x become very large.

We now apply this result to the theory of plane curves.

Suppose we have a curve of degree n. Writing its Cartesian equation in descending powers of x and y we obtain

$$(y-m_1x)(y-m_2x)\dots(y-m_nx)+ax^{n-1}+bx^{n-2}y+\dots+ky^{n-1}$$
$$+Ax^{n-2}+Bx^{n-3}y+\dots=0,$$

where $y-m_1x$, $y-m_2x$, ... are the factors of the terms of highest degree.*

The line $y=mx$ meets the curve where

$$(m-m_1)(m-m_2)\dots(m-m_n)\,x^n+(a+bm+\dots+km^{n-1})\,x^{n-1}$$
$$+\dots=0.$$

One root of this equation becomes very large, if the line $y=mx$ as it turns about the origin approaches one of the lines $y=m_1x,\dots,y=m_nx$; for then

$$(m-m_1)\;(m-m_2)\dots(m-m_n)\longrightarrow 0.$$

We may sum up this result in the convenient, if rather inaccurate, form

The terms of highest degree in the Cartesian equation of a curve equated to zero give the lines joining the origin to the points at infinity on the curve.

Let us now discuss a few cases in more detail.

First suppose m_1 not equal to any one of $m_2, m_3, \dots, m_n$.

The line $y=m_1x+c$ meets the curve where

$$\{p+(m_1-m_2)\dots(m_1-m_n)\,c\}\,x^{n-1}+(q+rc+\dots)\,x^{n-2}+\dots=0,$$

if we write

$$p\equiv a+bm_1+\dots+km_1^{\,n-1},\quad q\equiv A+Bm_1+\dots,$$
$$r\equiv b+\dots+(n-1)\,km_1^{\,n-2}.$$

Hence any line parallel to $y=m_1x$ meets the curve in one and only one infinite point,† with the exception of the line l for which $p+(m_1-m_2)(m_1-m_3)\dots(m_1-m_n)\,c=0$.

It follows at once that any projection of the curve touches the projection of l at the point where it crosses the vanishing line, i.e. l is an asymptote (Ch. I, § 4).

Next suppose $m_1=m_2\neq m_3, m_4, \dots$.

Now $y=m_1x+c$ meets the curve where

$$px^{n-1}+\{q+rc+(m_1-m_3)\dots(m_1-m_n)\,c^2\}\,x^{n-2}+\dots=0.$$

In general $y=m_1x+c$ meets the curve once and only once at infinity whatever finite value c may have, while the line at infinity meets the curve twice on $y=m_1x$. This evidently means that the line at infinity touches the curve* where it meets $y = m_1x$.

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* The reader will easily make the necessary modification in the argument if x is a factor of these terms.

† More accurately, 'meets the curve in exactly $n-1$ (finite) points'; but it is convenient to keep the convention 'every straight line meets a curve of degree n in n points'; and so throughout.

If, however, $y - m_1x$ is a factor of the terms of degree $n-1$ as well as of the terms of degree n in the equation of the curve, $p = 0$. Then every line parallel to $y = m_1x$ meets the curve in two points at infinity, while the two lines $y = m_1x + c$ for which $q + rc + (m_1 - m_3)\ldots(m_1 - m_n)c^2 = 0$ meet the curve in three points at infinity.

Hence the curve has a double point at infinity at which these two lines are tangents; i. e. these two lines are a pair of parallel asymptotes.

Suppose now $m_1 = m_2 = m_3 \neq m_4, m_5, \ldots$. If the line $y = m_1x + c$ meets the curve at only one infinite point for every finite value of c, the curve has an inflexion at infinity along $y = m_1x$ at which the line at infinity is the tangent.

If $y = m_1x + c$ meets the curve at two and only two infinite points for every finite value of c, the curve has a cusp along $y = m_1x$ at which the line at infinity is the tangent.

If $y = m_1x + c$ meets the curve at two and only two infinite points for all finite values of c but one, the curve has a double point at infinity along $y = m_1x$ at which the line at infinity is one tangent.

If $y = m_1x + c$ meets the curve at three and only three infinite points for all but three finite values of c, the curve has a triple point at infinity.

Similarly we may discuss the cases in which

$$m_1 = m_2 = m_3 = m_4 \neq m_5, m_6, \ldots;$$

and so on.

If a curve with a real equation passes through one circular point at infinity, it passes also through the other. For, if $y + ix$ is a factor of the terms of highest degree in the equation, so is $y - ix$. Such a curve is called a *circular* curve.

Similarly a curve with a real equation having a node at one circular point has a node at the other. Such a curve is called *bicircular*; and so on.

Ex. 1. $a^2y = x^3$ has a cusp at infinity, $ay^2 = x^3$ has an inflexion, $(a^2 - x^2)y = a^3$ and $3axy = x^3 + a^3$ have crunodes, $a^3y = x^4$ has a triple point, $ay^3 = x^4$ has an undulation.

Trace these curves.

[Use § 5. Another method is to obtain the equation of a projection of the curve by replacing x, y by ax/y, a^2/y as in Ch. I, § 3.]

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* More accurately: the vanishing line touches any projection of the curve.

Ex. 2. The equation of an n-ic with n distinct asymptotes

$$a_1x+b_1y+c_1=0,\ \dots,\ a_nx+b_ny+c_n=0$$

is of the form

$$(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)\dots(a_nx+b_ny+c_n)+u_{n-2}+u_{n-3}+\dots=0,$$

where u_k is homogeneous of degree k in x and y.

[Any one of the given lines meets the curve in only $n-2$ finite points.]

Ex. 3. Show that a cubic meets its three asymptotes in three finite collinear points Generalize by projection.

[The cubic is $f\equiv uvw+p=0$, where $u=0$, $v=0$, $w=0$ are the asymptotes and $p=0$ is the required line. On projection we get 'the tangents at three collinear points of a cubic meet the curve again in three collinear points'. See Ch. XII. § 4.]

Ex. 4. A variable cubic has a cusp and three fixed asymptotes. Show that the cusp lies on the conic touching at their middle points the sides of the triangle formed by the asymptotes.

[The conic is $\dfrac{\partial^2 f}{\partial x^2}\cdot\dfrac{\partial^2 f}{\partial y^2}=\left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2$ in Ex. 3.]

Ex 5. A quartic meets its four asymptotes in eight finite points lying on a conic.

Generalize by projection.

Ex. 6. If an n-ic has r asymptotes parallel to $y=mx$, the terms of the n^{th}, $(n-1)^{\text{th}}$, ..., $(n-r+1)^{\text{th}}$ degrees in its equation have respectively the factors $(y-mx)^r$, $(y-mx)^{r-1}$, ..., $(y-mx)$.

The equations of a circular and bicircular n-ic are of the forms

$$(x^2+y^2)\,v_{n-2}+u_{n-1}+u_{n-2}+\dots=0,$$

and

$$(x^2+y^2)^2\,v_{n-4}+(x^2+y^2)\,v_{n-3}+u_{n-2}+\dots=0.$$

Ex. 7. An n-ic has n distinct asymptotes all passing through O. Show that O is the centroid of the intersections of the n-ic with any line through O.

[Taking O as origin, the curve is $u_n+u_{n-2}+u_{n-3}+\dots+u_0=0$ Also $y=mx$ meets this curve in n points whose abscissae and ordinates have zero sum.]

Ex. 8. Two curves with distinct finite asymptotes meet in points whose centroid is the same as the centroid of the intersections of the asymptotes.

Ex. 9. A varying n-ic passing through n given points at infinity meets a given ellipse in $2n$ points whose eccentric angles have a constant sum (to within a multiple of 2π).

If the n points can be paired to form an involution with the points at infinity on the axes of the ellipse as double points, the sum is a multiple of 2π.

The sum of the eccentric angles of the intersections of a circle and ellipse is a multiple of 2π.

Ex. 10. If a real n-ic meets the line at infinity at the circular points only, any line through a fixed point O meets the curve in n points whose distances from O have a constant product.

Conversely, if the product is constant, the curve meets the line at infinity only at the circular points.

[Use Ch. I, § 6, Ex. 1. n must be even. Note the cases $n=2$ or 4.]

Ex. 11. The sum of the angles which the n asymptotes of the n-ics $u = 0$, $v = 0$ make with a given line is the same. Show that one curve of the pencil $u + kv = 0$ is circular.

Ex. 12. The lines joining a point P to n fixed points make angles with a fixed direction whose sum is α (plus a multiple of π). Show that the locus of P is an n-ic whose asymptotes are parallel to the sides of a regular polygon, one of whose sides makes an angle α/n with the fixed direction.

[The case $n = 2$ is well known.]

§ 6. Relations between Coefficients.

The information that a curve with equation $f(x, y) = 0$ has a node at a given point (x, y) is equivalent to assigning three linear relations between the coefficients of $f(x, y)$, namely

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

If (x, y) is stated to be a cusp, we have a fourth relation between the coefficients

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

Similarly the information that a curve has a k-ple point at a given point is equivalent to $\frac{1}{2}k(k+1)$ linear relations.

The information that the curve has a node (not at a *given* point) is equivalent to one relation between the coefficients namely the result of eliminating x, y from (i).

The information that the curve has a cusp is equivalent, to two relations found by eliminating x and y from (i) and (ii); and similarly that it should have a k-ple point to $\frac{1}{2}k(k+1) - 2$ relations.

We see that a curve whose equation is written down at random has no double point. Such a curve is called *non-singular*. It has no 'point-singularities', i. e. multiple points, though it has in general 'line-singularities', i. e. multiple tangents.*

The Cartesian equation of a curve of degree n contains one constant term, two terms of the first degree, three of the second degree, ..., $n+1$ of the nth degree. Hence the equation has $\frac{1}{2}(n+1)(n+2)$ coefficients.

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* The definition is therefore not free from objection. 'Anautotomic' (not cutting itself) which has been suggested would, however, not exclude acnodes or unreal multiple points.

One of these coefficients may be taken as unity without loss of generality. We have then

$$\tfrac{1}{2}(n+1)(n+2)-1=\tfrac{1}{2}n(n+3)$$

arbitrary coefficients. Hence

A curve of degree n is determined in general by $\frac{1}{2}n(n+3)$ *conditions.*

We mean by this that only a finite number of n-ics can be found to satisfy the given conditions, and that no n-ic can be found in general satisfying given conditions, if their number exceeds $\frac{1}{2}n(n+3)$.

If the conditions are that the curve is to pass through $\frac{1}{2}n(n+3)$ assigned points, we have $\frac{1}{2}n(n+3)$ *linear* relations between the coefficients of the curve's equation, which determine the ratios of these coefficients uniquely. Hence

One and only one curve of degree n can be found in general passing through $\frac{1}{2}n(n+3)$ *given points.*

We say 'in general'; for it may happen that the $\frac{1}{2}n(n+3)$ relations between the coefficients are inconsistent or not independent, and then the theorem is not true.

Consider, for instance, the case $n=2$. We have 'One and only one conic can be drawn through five points'; which is true in general. If three of the points are collinear, the conic is degenerate, being a line-pair. This is legitimate, for we did not exclude in the theorem the possibility of the n-ic splitting up into simpler curves.

But, if four of the points are collinear, an infinite number of conics pass through the points; for the line through these four points and any line whatever through the fifth point form such a conic.

As an easy deduction from the results of this section we have

A finite number of curves of degree n can in general be drawn having δ *nodes and* κ *cusps and satisfying*

$$\tfrac{1}{2}n(n+3)-\delta-2\kappa$$

other relations.

If the nodes and cusps are at assigned points, the number of other relations is $\frac{1}{2}n(n+3)-3\delta-4\kappa$.

For instance, if a curve has δ nodes, there are δ sets of values of x and y satisfying (i). Expressing the conditions for this, we have δ relations between the coefficients of $f(x, y)$.

But such results are not universally true, and must be applied with due care; as is shown by consideration of Ex. 12, 17 below, or of Chap. IV, § 7, Ex. 5 to 7.*

Ex. 1. Show that in general exactly one cubic can be drawn through 9 given points; but that through the 9 intersections of two given cubics a singly infinite number of cubics can be drawn.

[If $u = 0$, $v = 0$ are the two cubics, $u + kv = 0$ is any cubic through their intersection.]

Ex. 2. In general one n-ic can be drawn with a given node and passing through $\frac{1}{2}(n^2+3n-6)$ other given points; while two n-ics can be drawn with a given cusp and passing through $\frac{1}{2}(n^2+3n-8)$ other points.

Ex. 3. To be given a k-ple point and the tangents at that k-ple point is equivalent to being given $\frac{1}{2}k(k+3)$ linear relations between the coefficients of the curve's equation.

Ex. 4. The equation of any n-ic through r given points can be put in the form $S + a_1 S_1 + a_2 S_2 + \dots + a_\mu S_\mu = 0$, where $S, S_1, \dots, S_\mu$ are given polynomials such that $S = 0$, $S_1 = 0$, …, $S_\mu = 0$ are n-ics through the r given points, the a's are arbitrary constants, and $\mu = \frac{1}{2}n(n+3) - r$.
It is assumed that there is no identical relation of the form

$$S + b_1 S_1 + b_2 S_2 + \dots + b_\mu S_\mu \equiv 0.$$

Ex. 5. The equation of any n-ic with r given nodes is

$$S + a_1 S_1 + a_2 S_2 + \dots + a_\mu S_\mu = 0,$$

where $S = 0, \dots, S_\mu = 0$ are given n-ics with the r given nodes, and

$$\mu = \frac{1}{2}n(n+3) - 3r.$$

[Use § 4, Ex. 8. The result may be extended to cover the case of any given multiple points.]

Ex. 6. Obtain the equation of any n-ic with r given double points and given tangents at those double points.

[As in Ex. 5 with $\mu = \frac{1}{2}n(n+3) - 5r$.]

Ex. 7. The equation of any cubic with a given node is

$$(a_1 x + b_1 y + z)u^2 + (a_2 x + b_2 y + z)uv + (a_3 x + b_3 y + z)v^2 = 0,$$

where $u = 0$ and $v = 0$ are given lines through the node.

Ex. 8. The equation of any quartic with three given nodes is

$$a/u^2 + b/v^2 + c/w^2 + 2f/vw + 2g/wu + 2h/uv = 0,$$

where $u = 0$, $v = 0$, $w = 0$ are the sides of the triangle whose vertices are the nodes.

Ex. 9. Find the general equation of quintics with nodes at (1, 1, 1), (−1, 1, 1), (1, −1, 1), (1, 1, −1).

[$ax(x^2-y^2)(x^2-z^2) + by(y^2-z^2)(y^2-x^2) + cz(z^2-x^2)(z^2-y^2)$
$+ (l_1 y + m_1 z)(y^2-z^2)^2 + (l_2 z + m_2 x)(z^2-x^2)^2 + (l_3 x + m_3 y)(x^2-y^2)^2 = 0.$]

Ex. 10. The equation of any quintic with six given nodes 1, 2, 3, 4, 5, 6 is $aC_1 C_2 L_{12} + bC_3 C_4 L_{34} + cC_5 C_6 L_{56} = 0$; where $C_1 = 0$ is the conic through 2, 3, 4, 5, 6 and $L_{12} = 0$ is the line 12, &c.

[Use Ex. 5. The reader may find the general equation of quintics with nodes at $(\pm 2, 0)$, $(0, \pm 1)$, $x = \pm y = \infty$.]

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* The reader may consult a paper by J. E. Campbell in *Messenger Math.*, xxi (1892).

Ex. 11. The equation of any sextic with eight given nodes is

$$aU^2+2hUV+bV^2=CQ,$$

where $U=0$, $V=0$ are two cubics through the eight nodes, $C=0$ is a conic through five of the nodes, and $Q=0$ is the quartic through these five nodes having double points at the other three nodes.

Ex. 12. If a sextic has nine nodes of which eight are fixed, the ninth node lies on a certain 9-ic with a triple point at each of the eight fixed nodes.

[See Halphen, *Bull. de la Soc. Math. de France*, x (1882), p. 162; Hodgkinson, *Proc. London Math. Soc.*, II. xv (1916), p. 343.

Note that nine nodes of a sextic cannot be chosen arbitrarily; even though such a choice is equivalent to $\frac{1}{2}6(6+3)=27$ conditions.]

Ex. 13. The general equation of a sextic with seven given nodes is

$$au^2+bv^2+cw^2+2fvw+2gwu+2huv+dJ=0$$

where $u=0$, $v=0$, $w=0$ are cubics through the given nodes, and $J\equiv\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ is their Jacobian (Ch. VII, § 10).

[$J=0$ has a node at each of the seven points.]

Ex. 14. If in Ex. 13 $d=0$, all cubics through the seven nodes and a point P on the sextic pass through another point P' of the sextic. Each such cubic meets the sextic again in points Q and Q' such that all cubics through the nodes and Q pass through Q'.

[If u_1, v_1, w_1 are the values of u, v, w when the coordinates of P are substituted for x, y, z, the cubics through the seven nodes and P are $(wu_1-w_1u)+k(wv_1-w_1v)=0$. Hence at their ninth intersection $u:v:w=u_1:v_1:w_1$. Therefore, if P lies on the sextic, so does the ninth intersection. See Rohn, *Math. Annalen*, xxv, p. 598.]

Ex. 15. If $u=0$ is a conic and $v=0$ a cubic, $u^3=v^2$ is a sextic with cusps at $u=v=0$, the cuspidal tangents touching $v=0$.

[The preceding examples may suggest interesting investigations to the reader. For instance, is $u^3=v^2$ the most general sextic with six cusps? What is the most general septimic with ten given nodes?, &c.]

Ex. 16. The statement 'a curve has an inflexion at a given point' is equivalent to two relations between the coefficients in general.

Ex. 17. Show, however, that to be given three collinear inflexions of a cubic is equivalent to five (not six) conditions. For instance, show that a singly infinite family of cubics can be drawn with three given collinear inflexions and a given node.

[Taking (0, 0, 1) as node and (1, 0, 0), (0, 1, 0), (1, m, 0) as inflexions, the cubic is $xy(y-mx)+kz(mxy-m^2x^2-y^2)=0$, where k is an arbitrary constant.]

Ex. 18. The statement 'a curve has a tangent touching at k points' is equivalent to $k-2$ relations between the coefficients.

Ex. 19. If (x', y') is a centre of the n-ic $f(x,y)=0$, all the partial derivatives of f with respect to x, y of orders $n-1$, $n-3$, $n-5$, ... vanish when $x=x'$, $y=y'$.

[The curve comes to self-coincidence when rotated through 180° about a centre. If the centre is the origin, the terms of degrees $n-1$, $n-3$, $n-5$, ... in x and y vanish.]

Ex. 20. An n-ic passes through r given points and has a centre; r being $\frac{1}{4}(n+2)^2$ or $\frac{1}{4}(n+1)(n+3)$ as n is even or odd. Find the degree of the locus of the centre.

CHAPTER III

CURVE-TRACING

§ 1. The Object of Curve-tracing.

THE problem 'Trace a curve with given Cartesian equation is capable of more than one interpretation. It may mean that a mechanical construction is required; for instance, a circle can be traced by means of compasses. Or it may imply that the curve is to be drawn with the utmost possible degree of accuracy, finding for this purpose a large number of points on the curve. Or we may wish to obtain only a rough idea of the main features of the curve (e. g. the number of its branches, the position of its asymptotes and singular points, &c.), and to draw a sketch of the curve which gives some approximation to the truth.

By 'curve-tracing' we shall here mean the third of these alternatives; though it will be useful to indicate, where possible, how a more accurate diagram may be obtained; even if for most practical purposes a rough sketch of the curve will suffice, and we can spare ourselves the rather considerable expenditure of time which a more detailed tracing usually requires.

§ 2. The Method of Curve-tracing.

The following hints are useful in curve-tracing; the rectangular Cartesian equation of the curve being supposed given.

(i) Find where the curve meets the axes of reference.

(ii) See if the curve has symmetry. If only even powers of x occur in its equation, the curve is its own reflexion in $x = 0$. Similarly if only even powers of y occur. If the equation contains only terms of odd, or only terms of even degree, the curve is symmetrical about the origin, which is called a *centre* of the curve.

(iii) Notice if any values of x make y unreal (or vice versa). If so, these values of x do not correspond to any real part of the curve.

(iv) Equate to zero the terms of lowest degree in the equation to obtain the tangents at the origin, and equate to zero the terms of highest degree to obtain the lines joining the origin to the points at infinity on the curve. (Ch. II, §§ 3, 5.) Use Newton's diagram to get approximations to the shape of the curve near and very far from the origin (§ 3). If this fails to give any fresh information, proceed as in § 4.

(v) Find the asymptotes of the curve, and, if necessary, find on which side of an asymptote the curve approaches it at the two ends.

(vi) Find the finite intersections of the asymptotes with the curve: for, if the curve is of degree n, this requires the solution of an equation of degree $n-2$ at most.

(vii) Find if it is possible to get any number of points on the curve by solving equations of degree 1 or 2 at most.

The above hints are usually more than sufficient to obtain all the information necessary for tracing the curve, though occasionally special devices are needed.

It may sometimes be useful to obtain the tangent at one or more points of the curve. It is also well to remember that a tracing of an n-ic cannot be correct, if it is met by a line in r real points, where $n-r$ is negative or odd.

§ 3. Newton's Diagram.

Suppose that any term in the equation of a curve is $Ax^{\alpha}y^{\beta}$, A being a numerical coefficient, and α, β zero or positive integers. Consider a geometrical representation of the terms of the equation such that $Ax^{\alpha}y^{\beta}$ is represented by the point (α, β) referred to rectangular Cartesian axes placed as in Fig. 1. The points thus obtained are said to form *Newton's diagram* for the curve.

Suppose now that, when x and y are both small or both large, x^p and y^q are of the same order of magnitude Then the order of the term $Ax^{\alpha}y^{\beta}$ is the same as that of $y^{\beta+\alpha q/p}$, whose index is equal to the intercept made on the axis of y in Newton's diagram by a line through (α, β) parallel to the line making intercepts p and q on the axes of x and y.

Suppose that a line* joining two or more points of Newton's diagram (not parallel to an axis of reference) is such that all other points of the diagram lie to the right and above the line. Then, if the points on this line are considered as representing terms of the same order of magnitude, all the other points of the diagram represent terms of higher order of magnitude. If we suppress all these higher terms in the equation of the curve and then divide out by any power of x or y, which is a factor of the remaining terms, we get an approximation to the curve near the origin.

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* If the curve does not pass through the origin, there is no such line. If the curve does so pass, there may be one or more.

Similarly, if all points not on the line lie to the left and below the line, we get an approximation to the curve very far from the origin.

Illustrations of the use of Newton's diagram are given in §§ 4–8.

Ex. 1. Newton's diagram applies even if the original axes of reference are not rectangular.

Ex. 2. If a line through two or more points of Newton's diagram is such that all other points lie to the left and above or to the right and below the line, the terms represented by points on the line give an approximation at infinity.

For example, in the curve $x^2y + y^2 - 2xy - x = 0$ Newton's diagram gives the approximation $y^2 = x$ near the origin and $x^2 + y = 0$, $xy = 1$ far from the origin. As another example take the curve in § 7.

Ex. 3. Enunciate and prove a method similar to that of Newton's diagram for approximating in three dimensions to the shape of an algebraic surface near and far from the origin.

Apply the method to find the shape of a surface in the neighbourhood of a parabolic point.

[It approximates to the form of the surface $z = ay^2 + bx^3$ near the origin.]

§ 4. Examples of Curve-tracing.

In each of the following examples we shall only give details of the working when such details illustrate methods which have not been used in previous examples.

Ex. 1. Trace

$$x^3 - x^2y - 2xy^2 + 5xy + 2y^2 = 0.$$

The curve meets the axes of reference in no finite point except the origin. There is no symmetry.

The origin is a crunode at which the tangents are $y = 0$ and $5x + 2y = 0$. The lines joining the origin to the points at infinity are $x = 0$, $x = 2y$, $x + y = 0$.

Newton's diagram (Fig. 1) gives as approximations near the origin $x^2 + 5y = 0$, and $5x + 2y = 0$, and far from the origin

$$x(x - 2y)(x + y) = 0.$$

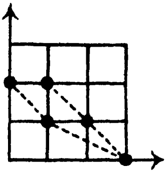

Fig. 1.

The only new information is that the branch through the origin touching $y = 0$ approximates to $x^2 + 5y = 0$ near the origin.

To get an approximation to the branch touching

$$5x+2y=0,$$

put $2y+5x=Y$. The curve becomes

$$0=Y^2-5xy-18x^3+9x^2Y-xY^2,$$

and Newton's diagram now gives the approximation

$$5Y+18x^2=0 \text{ or } 5(2y+5x)+18x^2=0.$$

Another method is to write the equation of the curve in the form $\quad 2y+5x=(-x^3+x^2y+2xy^2)/y.$

On the right-hand side of this equation put in the first approximation $-\frac{5}{2}x$ for y and we get the closer approximation $2y+5x=-\frac{18}{5}x^2$ as before. If the equation of the curve had contained terms of degree higher than the third, we should have retained only the terms of lowest degree in x in the numerator of the right-hand side after putting $-\frac{5}{2}x$ for y.

We know that there are asymptotes parallel to $x=0$, $x=2y$, $x+y=0$. These may be found as in Ch. II, § 5; or we may employ the device just used to find the approximation at the origin. Thus write the equation of the curve in the form

$$x-2y=\frac{-5xy-2y^2}{x(x+y)}.$$

On the right-hand side put in the first approximation $x=2y$, and we have the second approximation $x-2y=-2$, which is the asymptote. If the equation of the curve had contained terms of lower degree than the second, we should have retained only the terms of highest degree in y in the numerator of the right-hand side after putting $2y$ for x.

Similarly the other asymptotes are

$$x=1 \text{ and } x+y=1.$$

Write now the equation of the curve in the form

$$(x-1)(x-2y+2)(x+y-1)=-3x-4y+2;$$

where the left-hand side equated to zero is the equation of the asymptotes.

The curve evidently meets the asymptotes at their intersections with $\quad 3x+4y=2,$

giving three points on the curve. (See Ch. II, § 5, Ex. 3.)

We have now enough data to trace the curve (Fig. 2). We will verify the diagram by finding the side of each asymptote on which the curve approaches it at either end.

Write the curve

$$x-2y+2=\frac{-3x-4y+2}{(x-1)(x+y-1)}.$$

An approximation to the curve as it approaches the asymptote $x-2y+2=0$ is $x=2y-2$. Put this value of x into the right-hand side to get a closer approximation, and retain only the highest powers of y in the numerator and denominator. We get

$$x-2y+2=-\frac{5}{3y}.$$

For a given large positive value of y x is $2y-2$ for the asymptote, and $2y-2-\frac{5}{3y}$ for the curve; i.e. slightly smaller for the curve than the asymptote. Hence the curve

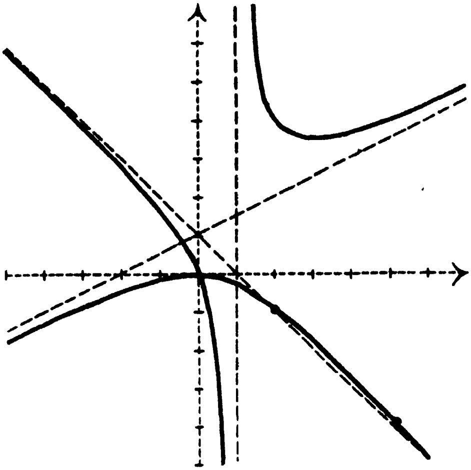

Fig. 2.

$x^3-x^2y-2xy^2+5xy+2y^2=0.$

lies close to the asymptote and to the *left* of it. Similarly, if y is large and negative, the curve lies to the *right* of the asymptote.

The reader may discuss the other asymptotes similarly.

The line $x=ty$ meets the curve where

$$x=\frac{(5t+2)}{(1+t)(2-t)},\quad y=\frac{5t+2}{t(1+t)(2-t)}.$$

Putting in any value for t we may obtain any number of points on the curve, and trace the curve accurately. As stated in § 1, we do not usually require this. However, the accurate tracing of most of the curves given in the examples of §§ 4 to 9 does not present much difficulty.

Ex. Trace the curves:

(i) $x^2y+2xy^2+6xy+2x^2+4x-4y=0.$

(ii) $xy(x^2-3xy+2y^2)+x^3-2x^2y-2xy^2+2y^3=5xy+x+2y.$

(iii) $3x(x^2+y^2)+2x^2+6xy+6y^2=4x.$

(iv) $9(x+4)y^2+24xy+x(x^2+2x-16)=0.$

(v) $x^3+y^3=(y-x)(y-2x).$

(vi) $x(x^2-xy+4y^2)+4y(2x-y)=0.$

(vii) $64y^2(x-4)+16x^2y+x^2(5x-12)=0.$

(viii) $2y^2(x-1)-2y(x^2-2x)+x^2(x-2)=0.$

(ix) $x^2y-y^2x+x^2-4y^2=0.$

(x) $xy(x^2+y^2)+x^3+y^3=x^2-y^2.$

(xi) $x(x^2-y^2)=4x^2-6xy-4y^2.$

(xii) $x(x+2y)(2x-y)=4x^2-xy+2y^2.$

(xiii) $(x+y)(x^2+y^2)+2x(x-y)=0.$

(xiv) $x(2x-y)(x+y)+3y^2=0.$

(xv) $y^4-x^4=a^2xy.$

[(i) The curve is $x(y+2)(x+2y+2)=4y$, showing that the asymptotes are $x=0$, $y+2=0$, $x+2y+2=0$, and meet the curve again where $y=0$.

The asymptote $y+2=0$ meets the curve in *no* finite point. It is an inflexional tangent at infinity, and therefore the curve approaches it on the same side at both ends.

(ii) $(x+1)(y+1)(x-2y-1)(x-y+1)=x^2+y^2-1.$

The asymptotes meet the curve on the circle $x^2+y^2=1$.

(iii) Writing the curve $y(x+2)=-x\pm\{-x(x^2-1)(x+\frac{8}{3})\}^{\frac{1}{2}}$, we see that the real asymptote is $x+2=0$ and that the tangent is parallel to $x=0$ at the points $(0,0)$, $(1,-\frac{1}{3})$, $(-1,1)$, $(-\frac{8}{3},-4)$.

For real points x must lie between 0 and 1 or between $-\frac{8}{3}$ and -1.

(iv) $3y(x+4)=-4x\pm\{-x(x+2)(x-4)(x+8)\}^{\frac{1}{2}}$. See Ch. XIV, Fig. 1.

(v) $(x+y-2)(x^2-xy+y^2)+xy+y^2=0.$

(vi) Real asymptote $x=1$. See Ch. XIII, Figs. 3, 4 for (vi) and (vii).

(vii) Real asymptote $x=4$.

(viii) Real asymptote $x=1$. See Ch. XIII, Fig. 2.

(ix) $(x+4)(y+1)(x-y-3)=x-16y-12.$

(x) $(x+1)(y+1)(x^2+y^2)=xy(x+y)+2x^2.$

Since $x+y$ is a factor of the terms of the third degree as well as of the terms of the second degree, $x+y=0$ is an inflexional tangent at the origin. (Ch. II, § 2.) An approximation at the origin is $x+y=-x^3$; the other approximation being $x-y=x^2$.

(xi) $(x+y-3)(x-y+3)(x-4)+9x+24y-36=0.$

(xii) $(x+1)(x+2y-2)(2x-y-2)+2(x+y-2)=0.$

(xiii) $(x^2+y^2-2y+1)(x+y+2)=x-3y+2.$

(xiv) $(x-3)(2x-y+4)(x+y+1)+14x+9y+12=0.$

(xv) Asymptotes $x=y$, $x=-y$. There is an inflexion on each branch at the origin. The curve has the origin as a centre of symmetry.]

§ 5.

Ex. II. Trace

$$(x^2-y^2)^2-(x^2-y^2)(x+3y)-3x^2+6xy+y^2=0.$$

Since $(x-y)^2$ is a factor of the terms of highest degree in this equation, the curve either has an infinitely distant tangent with its point of contact on $x=y$, or else has an infinitely distant double point on $x=y$.

We find that $x=y+c$ meets the curve where

$$4(c-1)^2y^2+2c^2(2c-3)y+c^2(c^2-c-3)=0;$$

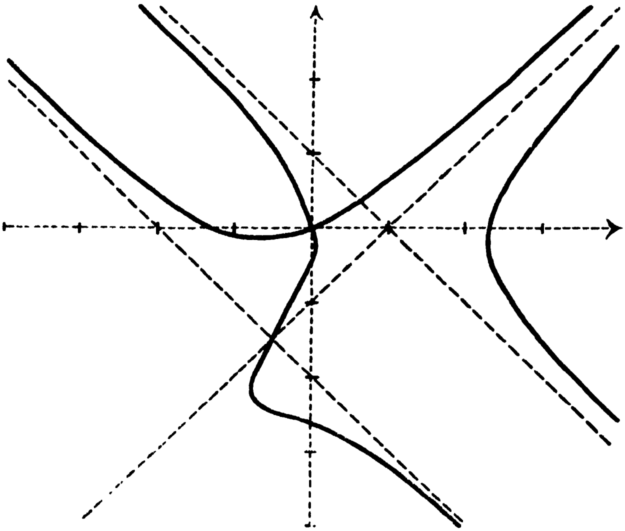

Fig. 3.

$(x^2-y^2)^2-(x^2-y^2)(x+3y)-3x^2+6xy+y^2=0.$

giving only two finite values of y, so that the latter alternative is correct. (See Ch. II, § 5.) The line $x=y+c$ meets the curve in only one finite point if $(c-1)^2=0$, so that the tangents at the infinite double point both coincide with $x-y=1$. Hence the double point is a cusp, and the curve has two asymptotes coinciding with $x-y=1$.

Similarly the curve has an infinite double point on $x+y=0$, the tangents at which are $x+y=1$ and $x+y=-2$. The curve has these lines as parallel asymptotes.

Writing the curve in the form

$$(x-y-1)^2(x+y-1)(x+y+2)=5x-3y-2$$

we see, as in § 4, that the curve meets its asymptotes where they meet $5x = 3y+2$; and that closer approximations to the form of the curve far from the origin are

$$x+y-1 = -2/3y, \quad x+y+2 = 1/2y, \quad (x-y-1)^2 = 1/2y.$$

The third of these approximations shows that the curve approaches the asymptote $x = y+1$ from both sides at one end only, namely the end at which y is positive. The other two approximations are interpreted as in § 4.

The origin is a crunode at which the tangents are

$$3x^2 - 6xy - y^2 = 0.$$

See Fig. 3.

The coordinates of any point on the curve may be put in the form

$$x = \frac{(t^2+2t-2)(t^2+t-1)}{2(t^2+t-2)}, \quad y = \frac{(t^2+2t-2)(t^2+t-3)}{2(t^2+t-2)}.$$

(See Ch. X, § 4, Ex. 3.)

Ex. Trace the curves:

(i) $x^3+2x^2y+xy^2-x^2-xy+2 = 0.$

(ii) $x^2y^2+xy+x+3y+3 = 0.$

(iii) $(x^2-y^2)^2+(x^2-y^2)(5y-x)-3x^2-2xy+9y^2 = 0.$

(iv) $xy(2xy-5x-10y)+50(x-y)^2 = 0.$

(v) $x(x-y)^2-2y(x-y)+y = 0.$

(vi) $x^2(x^2-y^2)+(x+1)(2x^2+xy-y^2)+x = 0.$

(vii) $x^2(2x-y)+x^2+x+y = 0.$

(viii) $x^2(x-y)^2-x(x-y)(3x+y) = 3x^2-8xy+2y^2.$

(ix) $y(x-y)^2 = x+y.$

(x) $xy(x+y)^2+(x+y)(x^2-3xy-y^2)-4x^2+2y^2 = 0.$

(xi) $x^2y^2+12a^3(3x+2y) = 0.$

(xii) $(x^2-1)(x-2)y^2-x^3+4y = 0.$

(xiii) $(x+y)x^2y^2+(x^2+6xy-y^2)xy+2(x^3+y^3) = 0.$

(xiv) $x^2y^2-2xy(x-2y)-3x^2+8xy+3y^2 = 0.$

(xv) $100x^2y^2+2y(100+90x)-144x^2+36x+469 = 0.$

(xvi) $10(x^2-4)(y^2-1)+xy+23x+26y+94 = 0.$

(xvii) $9x^2y^2+96xy+144(x+y)+496 = 0.$

(xviii) $x^2y^2-9x^2-y^2+25 = 0.$

(xix) $225(yx^2+1)^2 = (5+x)(1+x)(1+2x)(3-5x)(3-4x)(5-x).$

[(i) $x(x+y)(x+y-1)+2 = 0.$ The asymptote $x = 0$ is an inflexional tangent at infinity; compare § 4, Ex. (i).

(ii) Solving the equations as a quadratic in x or y we see that the tangents are parallel to $x = 0$ at $(-3, 0)$, $(1, -2)$, $(-\frac{3}{4}, -2)$ and parallel to $y = 0$ at $(0, -1)$, $(-6, \frac{1}{3})$, $(-6, -\frac{1}{4})$. The asymptotes are $x^2y^2 = 0$. See Ch. II, § 4, Ex. 2 (iii).

(iii) $(x-y+1)^2(x+y-1)(x+y-2) = x-7y+2.$

(iv) The curve has no real asymptote. The tangents are parallel to the axes of x and y at $(5, 10)$ and $(8, \frac{40}{7})$ respectively. Compare (ii).

(v) $(x+2)(x-y-1)^2+3x-3y-2=0$. There is an inflexion at the origin.

(vi) $(x^2+x+1)(x-y+1)(x+y)=y$.

(vii) $(x-1)(x+1)(2x-y+1)+3x+1=0$.

(viii) $(x-1)(x+2)(x-y-1)(x-y-3)=11x-8y-6$.

(ix) $y\{(x-y)^2-2\}=x-y$. The curve is symmetrical about the origin and has an inflexion at the origin.

(x) $(x-1)(y+1)(x+y-1)(x+y-2)=5x+y-2$.

(xi) The asymptotes are $x^2y^2=0$.

(xii) $(x^2-1)(x-2)(y^2-1)=2x^2+x-4y-2$. The curve has a crunode at $(\infty, 0)$, and a triple point at $(0, \infty)$. The intersections with the asymptotes and with $2x=3$, $y=2$ should be found.

(xiii) The curve has acnodes at $(0, \infty)$, $(\infty, 0)$ and a triple point at $(0, 0)$. It has a branch asymptotic to $x+y+6=0$ and an oval lying between the asymptote and the axes of reference. Consider its intersections with $y=tx$ or $txy+x+y=0$.

(xiv) $(x+1)(x+3)(y+1)(y-3)+16xy+12x+6y+9=0$.

(xv) Asymptotes $x^2(25y^2-36)=0$. Solve for x and y. See Ch. XVIII, Fig. 6.

(xvi) Asymptotes $x=\pm 2$, $y=\pm 1$.

(xvii) Asymptotes $x^2y^2=0$. Solve for x and y.

(xviii) Asymptotes $x=\pm 1$, $y=\pm 3$. Biflecnodes at $(0, \infty)$ and $(\infty, 0)$. Bitangents $x=y$, $9x=y$, $3x+y=4$, &c.

(xix) Asymptotes $x^2=0$. Solve for y. See Ch. XVIII, Fig. 2.

§ 6.

Ex. III. Trace

$$y^2(x+y)^2=x^2(y+2).$$

Newton's diagram (Fig. 4) gives us as the approximation at the origin

$$y^4=2x^2,$$

which is the pair of parabolas

$$y^2=\pm\sqrt{2}x.$$

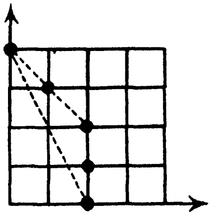

Fig. 4.

Hence there are two branches of the curve touching each other and touching the axis of y at the origin. The origin is called a *tacnode* in such a case. It is sometimes called also a 'double cusp'; but this nomenclature is open to objection, as we shall see later.*

As in § 5 we find the parallel asymptotes

$$y+1=0 \text{ and } y-2=0.$$

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* A tacnode is the point of contact of two ordinary branches of a curve.

There are two infinite points on the curve along $x+y=0$, since $(x+y)^2$ is a factor of the terms of the highest degree; but $x+y=c$ meets the curve at only one infinite point for all finite values of c. Hence the curve has an infinitely distant tangent with its point of contact on $x+y=0$. (Ch. II, § 5.) To get an approximation to the curve far from the origin along $x+y=0$, write its equation in the form

$$(x+y)^2 = x^2(y+2)/y^2.$$

Now put $-y$ for x on the right-hand side and retain the highest power of x in the numerator. We get as a closer approximation $(x+y)^2 = y$. This is a parabola with axis parallel to $x+y=0$, touching $y=0$ at the origin, and lying above the axis of x.

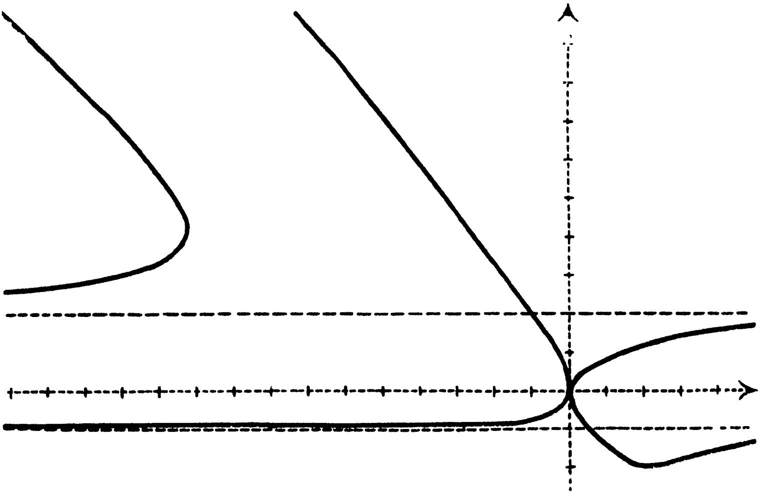

Fig. 5.

$y^2(x+y)^2 = x^2(y+2)$.

Writing the equation of the curve in the form

$$(y+1)(y-2)(x+y)^2 + (y^2+2y)(2x+y) = 0,$$

we see that it meets its asymptotes where $2x+y=0$, and readily find the side on which the curve approaches either asymptote.

Solving the equation as a quadratic in x we see that for real points on the curve $y \geqslant -2$. See Fig. 5.

Ex. Trace the curves:

(i) $(x+y)(x-y)^2 = 3xy - y^2$.

(ii) $x^2y = x^2 + y^2$.

(iii) $x^2y = y^2 + 2xy + 4x$.

(iv) $4y^2 = x^2(x - 2y)$.

(v) $(y + x)(y - 2x)^2 = 9xy$.

(vi) $x^2(x - y)^3 - x^2(x - y)^2 + y = 0$.

(vii) $x(y - x)^2 = a(y^2 + a^2)$.

(viii) $axy + a^3 = x^3$.

(ix) $y^2(x + y)^2 + ax(x + y)^2 + a^2xy = 0$.

(x) $x(y - x)^2(y + x)^2 + (y - x)^3(x + 2y) = 8x^2$.

(xi) $(xy + 6)^2 = 9(2y + 3)$.

(xii) $(xy + 4)^2 = 2(2y + 3)(y + 2)$.

(xiii) $x^2y^2 = 4(y + 1)$.

(xiv) $x^2y^2 = 4(y + 4)(y - 1)$.

(xv) $x^2y^2 = 16(y - 1)$.

(xvi) $16(3y + x^2)^2 = 7(x + 3)(x + 2)(x - 1)(x - 4)$.

[(i) $(x + y + 1)(x - y)^2 = x(x + y)$. Parabolic approximation $(x - y)^2 = x$.

(ii) Asymptote $y = 1$. Parabolic approximation $x^2 = y$. Symmetrical about $x = 0$.

(iii) Asymptote $y = 0$. Parabolic approximation $x^2 = y$. Tangents parallel to axes of x and y at $(-1, -1)$ and $(4, 4)$ respectively.

(iv) Asymptote $x = 2y + 1$. Parabolic approximation $2y = -x^2$.

(v) Asymptote $x + y + 1 = 0$. Parabolic approximation $(y - 2x)^2 = 6x$.

(vi) $x^2(x - y)^2(x - y - 1) = -y$. The approximations at infinity are $x^2 = 1/y^2$, $(x - y)^2 = 1/y$, $x - y - 1 = -1/y$. There is a tacnode at infinity at which $x = 0$ is the tangent, and a triple point at infinity at which $x = y$, $x = y$, $x = y + 1$ are the tangents. The tangent at $(1, 0)$ meets the curve again at $(1, 2)$.

(vii) Asymptote $x = a$. Parabolic approximation $(y - x)^2 = ay$

(viii) Asymptote $x = 0$. Parabolic approximation $x^2 = ay$.

(ix) $y^2(x + y - a)(x + y + a) + a(x + y)(x^2 + xy + ay) = 0$. Parabolic approximation $y^2 + ax = 0$. Approximations at origin $x^2 + ay = 0$ and $y^3 + ax = 0$. The asymptote $x + y = a$ is an inflexional tangent at infinity.

(x) A tacnode at the origin. Asymptotes $x = -2$, $x = y$ twice. The curve has an infinite cusp. Parabolic approximation $(x + y)^2 = 2y$.

(xi) Tacnode at $(\infty, 0)$, cusp at $(0, \infty)$. Asymptotes at $x^2y^2 = 0$. Solve for x and y.

(xii) Tacnode at $(\infty, 0)$, node at $(0, \infty)$. Asymptotes $y^2(x^2 - 4) = 0$. Solve for x and y.

(xiii) Tacnode at $(\infty, 0)$, cusp at $(0, \infty)$. Asymptotes $x^2y^2 = 0$.

(xiv), (xv) Tacnode at $(\infty, 0)$ at which the tangents are $y = 0$, but which is an isolated point on the curve. One curve has a node at $(0, \infty)$, and the other a cusp.

(xvi) Tacnode at $(0, \infty)$ with line at infinity as tangents. Solve for y. See Ch. XVIII, § 14.]

§ 7.

Ex. IV. Trace

$$y^2 = 2x^3 + xy^3.$$

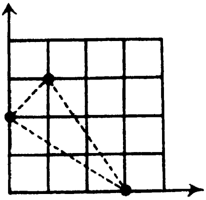

Fig. 6.

Newton's diagram (Fig. 6) gives $y^2 = 2x^3$ as the approximation at the origin and $2x^2 + y^3 = 0$ as an approximation at infinity. The asymptote is $x = 0$.

The curve has an inflexion at infinity with an infinitely distant tangent. (Ch. II, § 5, Ex. 1.)

The tangents parallel to the axes of reference are readily found.

The line $y = tx$ meets the curve where

$$x = t^{-3}\{-1 \pm (t^5+1)^{\frac{1}{2}}\}, \quad y = t^{-2}\{-1 \pm (t^5+1)^{\frac{1}{2}}\}.$$

Putting in values of t we get any number of points on the curve (Fig. 7). In particular, taking $t = -1$, we see that $x = -y$ touches the curve at $(1, -1)$.

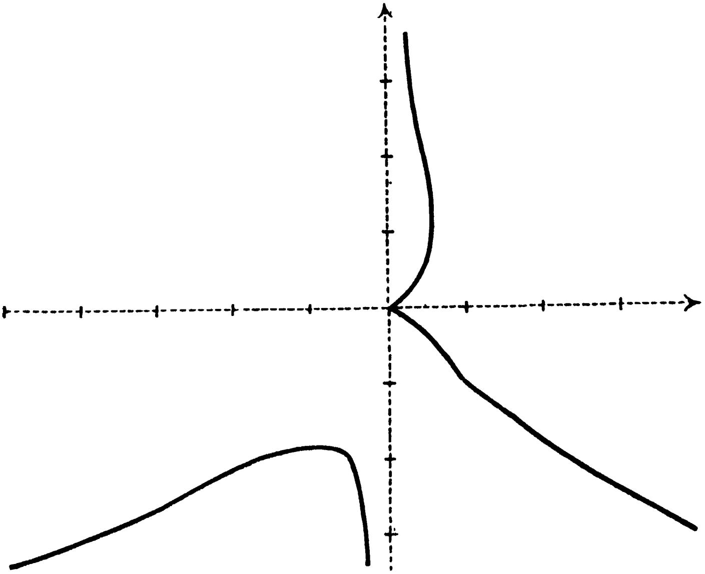

Fig. 7.
$y^2 = 2x^3 + xy^3$.

We may verify our approximations at infinity by writing down the equation of a projection as in Ch. I, § 3. Thus, if we replace x and y by x/y and $1/y$, the equation of the curve becomes $y^2 = 2x^3y+x$. This is the equation of the projection in which the axis of y is unaltered, while the line at infinity is projected into the axis of x, and vice versa. The projection has $x = 0$ as an ordinary tangent at the origin, and therefore the original curve has $x = 0$ as an ordinary asymptote.

Again, replacing x and y by $1/x$ and y/x, the equation becomes $x^2y^2 = 2x+y^3$. This is the equation of the projection in which the axis of x is unaltered, while the line at infinity is projected into the axis of y, and vice versa. The projection has $x = 0$ as an inflexional tangent at the origin, and therefore the original curve has the line at infinity as an inflexional tangent at $(\infty, 0)$.

The reader may apply this process to any of the examples in this chapter.

Ex 1. Trace the curves

(i) $x^2-3xy^2+2y^4+y^5 = 0$.

(ii) $x^4+a(x^3+y^3) = 3a^2xy$.

(iii) $x^3y-x^3+y^2 = 0$.

(iv) $x^3+(x-y)^2 = y$.

(v) $y^2-y = x^3-x$.

(vi) $x(x+y) = y^3-y^4$.

(vii) $x^3(x+y) = y^3+y^2$.

(viii) $y^2(x-y) = x^3(x+y)$.

(ix) $y^4+xy^3+x^2y^2+x^4y+x^6 = x^2y^3$.

(x) $xy+y^4+x^2y^3 = x^3$.

(xi) $x^3(x-2y) = 8y^2(y-x)-4(y^2-x^2)$.

(xii) $y^3(x-y)^2+x^2y = x^4+y^4$.

(xiii) $y^3(x-y)^2+x^4+y^4+2x^2(x-y)+x(x+2y) = 0$.

(xiv) $x^3y^2 = (y-x^2)(y-2x^2)(y-3x^2)$.

(xv) $3x^4 = 2x^2y(x+1)+y^2$.

(xvi) $y^4+x^3(x^2-y^2) = 0$.

(xvii) $y^2-2yx^3+x^7 = 0$.

(i) Approximations at origin $y^2 = x$ and $2y^2 = x$; at infinity $x^2+y^5 = 0$.

(ii) Approximations at origin $3ay = x^2$ and $3ax = y^2$; at infinity $x^4+ay^3 = 0$.

(iii) Approximation at origin $y^2 = x^3$; at infinity $x^3+y = 0$. The asymptote $y = 1$ is tangent at an infinite point of undulation.

(iv) Approximation at origin $y = x^2$; at infinity $y^2 = -x^3$.

(v) Approximation at origin $x = y-y^2$; at infinity $y^2 = x^3$. The curve is symmetrical about $2y = 1$.

(vi) The curve is closed. The tangents are parallel to the axes of x and y when $y = \frac{1}{2}(1 \pm \sqrt{2})$ and when $x = -1, \frac{1}{8}, \frac{1}{8}$ respectively.

(vii) Approximation at origin $y = \pm x^2$; at infinity $y^2 = x^3$. The asymptote is $x+y+1 = 0$ meeting the curve at $(0, -1)$ and $(-\frac{1}{2}, -\frac{1}{2})$. The line $y+1 = 0$ is an inflexional tangent at $(0, -1)$ and meets the curve again at $(1, -1)$, where the tangent is $x = 1$ which meets the curve again at $(1, 1)$.

(viii) Approximations at origin $y^2 = x^3$ and $x+y = 2x^2$; at infinity $y^2 = -x^3$ and asymptote $x+y = 2$.

(ix) Approximations at infinity $y = x^2$ and $y^3 = x^4$. An isolated quadruple point at the origin.

(x) Approximations at origin $x = -y^3$ and $y = x^2$; at infinity $x = y^3$ and $y = -x^2$.

(xi) Approximations at origin $8(y-x) = x^3$ and $y+x = 2y^2$; at infinity $x^3+4y^2 = 0$. The asymptote $x-2y+1 = 0$ meets curve at $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{3}{4})$.

(xii) Approximations at origin $y = x^2$ and $x^2 = y^3$; at infinity $x^2 = y^3$ and $(x-y)^2 = 2x$.

(xiii) Approximations at origin $x+2y = -3x^2$ and $y^3 = -2x$; at infinity $y^3 = -x^2$ and $(x-y)^2 = -2x$. The curve has a crunode at the origin, touches $y = 0$ at $(-1, 0)$, and passes through $(0, -1)$.

(xiv) Approximations at origin $y = x^2$ and $y = 2x^2$ and $y = 3x^2$; at infinity $y = x^3$ and $y^2 = -6x^3$. Any number of points may be found by considering the intersections of the curve with $y = tx^2$. (Take especially the cases $t = \frac{3}{2}$ and $t = \frac{5}{2}$.)

(xv) Approximations at origin $y = x^2$ and $y = -3x^2$; at infinity $y = -2x^3$ and $2y = 3(x-1)$. Consider the intersections with $y = tx^2$.

(xvi) Asymptotes $\pm 2y = 2x+1$. Approximation at origin $y^4+x^5 = 0$: at infinity $y^2 = x^3$.

(xvii) Approximations at origin $y = 2x^3$ and $2y = x^4$; at infinity $y^2+x^7 = 0$.]

Ex. 2. Trace $x^n+y^n = nax^{n-1}$ when n is an odd or even positive or negative integer.

Ex. 3. Trace $x^n+y^n = n^2a^2x^{n-2}$ when n is integral.

Ex. 4. Trace $x^{2n+1}+y^{2n+1} = (2n+1)ax^ny^n$ when n is integral.

Ex. 5. Trace $(x^2-a^2)^2+(y^2-b^2)^2 = c^4$, distinguishing the cases $a>b>c$, $a>b=c$, $a>c>b$, $a=c>b$, $(a^4+b^4)^{\frac{1}{4}}>c>a>b$, $(a^4+b^4)^{\frac{1}{4}}=c>a>b$, $a=b=c$.

§ 8.

Ex. V. Trace

$$x^4-2x^3y+2x^2y+y^2 = 0.$$

Newton's diagram (Fig. 8) gives as an approximation at infinity $2x^3 = y$. Hence the curve has a cusp at infinity with an infinitely distant cuspidal tangent.

There is an asymptote $x-2y+1 = 0$.

Newton's diagram gives $(y+x^2)^2 = 0$ as the approximation at the origin, namely two coincident parabolas. This indicates that the origin is a somewhat complex singularity on the curve. To ascertain more closely the nature of the curve at the origin, write its equation in the form

$$y = -x^2 + x^3 \pm \sqrt{(-2x^5 + x^6)}.$$

We see that for real points on the curve x must not lie between 0 and 2. Hence the approximation at the origin is given only by those parts of the two coincident parabolas for which x is negative (Fig. 9).

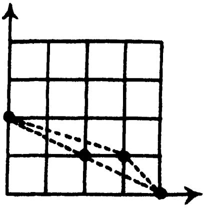

Fig. 8.

The origin is sometimes called in such cases a 'rhamphoid (ramphoid) cusp'* or 'cusp of the second species', the ordinary cusp being 'a ceratoid cusp'† or 'a cusp of the first species'.

The nomenclature is objectionable, for it implies that cusps of the first and second species are comparable, whereas the former is the cusp of the simplest possible type and the latter is really a singularity of considerable complexity.

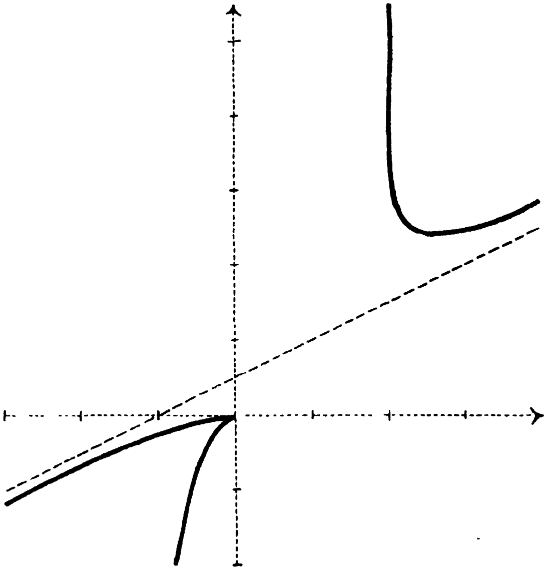

Fig. 9.

$x^4 - 2x^3y + 2x^2y + y^2 = 0.$

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* From ῥάμφος – beak.

† From κεράτιον = a little horn.

Ex. 1. Trace the curves

(i) $x^3(x+y)^2+x^2(x^2+2xy-y^2)+xy(2x-y)+y^2=0$.

(ii) $(y^3+x^2)^2=y^2x^3$.

(iii) $(5y+x^2)^2=4x(x-3)(x+5)$.

(iv) $(y+x^2)^2=4x(x^2-2x+2)$.

(v) $4(xy+1)^2=x(9x+16)$.

(vi) $4(xy+1)^2=x(25x+16)$.

(vii) $(2y-x^2)^2+4xy(y+x^2)=0$.

(viii) $(2y+x^2)^2=x$.

[(i) The asymptotes are $(x+1)(x-1)^2=0$, and there is a parabolic approximation at infinity $(x+y)^2=-2x$. There is a 'rhamphoid cusp' at the origin. The tangents at $(-\frac{1}{2}, -\frac{1}{8})$ and $(-2, \frac{4}{3})$ are parallel to $x=0$; as is seen by solving for y.

(ii) The approximations at infinity are $y^2=x$ and $y^4=x^3$. The approximation at the origin is that part of $(y^3+x^2)^2=0$ for which $x>0$. The curve meets $x^2=ty^3$ where $x=t^{-4}(t+1)^6$, $y=t^{-3}(t+1)^4$.

(iii) Rhamphoid cusp at $(0, \infty)$, See Ch. XVIII, § 15.

(iv) Rhamphoid cusp at $(0, \infty)$. The line $x+y=2$ touches at $(2, 0)$ and $(1, 1)$.

(v) Rhamphoid cusp at $(0, \infty)$. Asymptotes $x^2(4y^2-9)=0$.

(vi) Rhamphoid cusp at $(0, \infty)$. Asymptotes $x^2(4y^2-25)=0$.

(vii) Rhamphoid cusp at origin. Node at $(0, \infty)$ with tangents $x+1=0$ and the line at infinity. Also asymptote $4x+16y+5=0$.

(viii) There is a singularity at $(0, \infty)$ whose projection is like a rhamphoid cusp in shape.]

Ex. 2. If u_r, v_r, and w_r are homogeneous of degree r in x and y, the curves

$$0=u_1^2+u_3+u_4+\dots+u_n,$$
$$0=u_1(u_1+u_2)+u_4+\dots+u_n,$$
$$0=(u_1+v_1^2)^2+u_1^2w_1+u_1u_3+u_5+\dots+u_n,$$

have in general an ordinary (keratoid) cusp, a tacnode, a rhamphoid cusp respectively at the origin.

[Choose axes of reference such that $u_1\equiv y$, $v_1\equiv x$. Then use Newton's diagram.]

Ex. 3. The tangent at an ordinary cusp, a tacnode, a rhamphoid cusp meets the curve in 3, 4, 4 points respectively coinciding with the double point.

Ex. 4. The radii of curvature of the two branches at an ordinary cusp, a tacnode, a rhamphoid cusp are respectively both zero, non-zero, and in general unequal, non-zero, and equal.

[The radius of curvature at the origin of a curve touching $y=0$ there is the limit of $x^2/2y$ as we approach the origin.]

Ex. 5. The ratio of the radii of curvature of the two branches of a curve at a tacnode is unaltered by projection.

[See Ch. I, § 6, Ex. 6. Note the case in which these radii of curvature are equal and opposite, as in Fig. 5.]

Ex. 6. The curve

$$0 = (u_1+v_1^2)^2+(u_1 w_1+u_3)(u_1+v_1^2)+u_1^2 u_2+u_1 u_4+u_5+\ldots+u_n$$

has two ordinary branches touching $u_1 = 0$ at the origin and having the same curvature there.

[The origin is called an 'oscnode'. Contrast the case of the rhamphoid cusp in Ex. 2; and see Ch. XVII, § 8 (iv).

Making the equations of the curves in Ex. 2 and 6 homogeneous by means of $u_1 = v_2$, we get the lines joining the origin to the intersections of the curve and $u_1 = v_2$. If we choose the coefficients of v_2 so as to make as many of these lines as possible coincide with $u_1 = 0$, $u_1 = v_2$ becomes a conic of closest contact with the curve at the origin. A doubly infinite number of conics meet a curve five times at a rhamphoid cusp, and no conic meets it six times there.]

Ex. 7. To give a tacnode or rhamphoid cusp and the tangent at that point is equivalent to assigning 6 or 7 relations respectively between the coefficients of the equation of a curve.

If the tangent is not given, the number of relations is one less.

§ 9.

If the coordinates of a point on a curve are given in terms of a parameter t by means of the equations

$$x = f(t), \quad y = \phi(t),$$

we may obtain the equation of the curve by eliminating t between these two equations and then trace it by the methods of §§ 4–8.

As an alternative we may obtain any number of points on the curve by taking various values for t and so trace the curve. But this is apt to be laborious, and the method of the following example will often suffice.

Ex. VI. Trace

$$x = (1+t)(1-2t)/t, \quad y = (1+t)^2(1-2t)^2/t.$$

The curve only meets the axes at the origin, and then $t = -1$ or $\frac{1}{2}$. We have $y = tx^2$, and therefore the approximations at the origin are $y = -x^2$ and $2y = x^2$.

Again, x and y are infinite when $t = 0$ or ∞.

Since $y/x = (1+t)(1-2t)$, we see that there is an asymptote parallel to $y/x = 1$ corresponding to $t = 0$.

Because $x-y = (1+t)(1-4t^2)$, we obtain $x-y = 1$ as the asymptote, on putting $t = 0$. The curve meets the asymptote where $(1+t)(1-4t^2) = 1$, whence $t = 0$ or $\frac{1}{2}(\pm\sqrt{2}-1)$, i. e. $x = \infty$ or $2\pm\sqrt{2}$.

Moreover, putting $t = \infty$ in

$$x^3/y = (1+t)(1-2t)/t^2,$$

we see that $x^3+2y=0$ is the other approximation at infinity, (see Fig. 10).

The tangents parallel to $x=0$ are given by $\dfrac{dx}{dt}=0$ and are unreal. Similarly, the only real tangents parallel to $y=0$ are the tangents at the origin.

As other examples the reader may take the curves in §§ 4 and 5, or those given in Ex. 2 below.

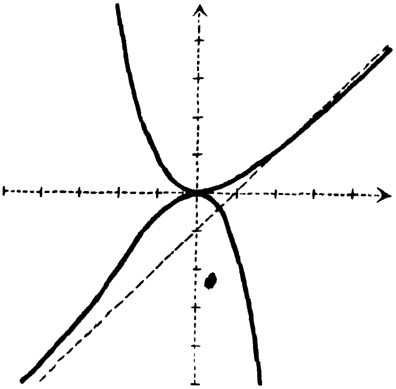

Fig. 10.
$2y^2+x^2y(x+1)=x^4$.

Ex. 1. The real ordinary cusps of a curve are given by values of the parameter t making $\dot{x}$ and $\dot{y}$ both zero, dots denoting differentiation with respect to t. The inflexions are given by $\dot{x}\ddot{y}=\ddot{x}\dot{y}$.

[The radius of curvature is $(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}\div(\dot{x}\ddot{y}-\ddot{x}\dot{y})$. It vanishes at an ordinary cusp (see § 8, Ex. 4), and is infinite at an inflexion.]

Ex. 2. Trace the curves

(i) $x=3at^2/(1+t^3),\quad y=3at/(1+t^3)$.
(ii) $x=5at^2/(1+t^5),\quad y=5at^3/(1+t^5)$.
(iii) $x=(t^4+8t),\quad y=(t^3+1)$.
(iv) $x=(t^4-2t^2),\quad y=t^3-3t$.
(v) $x=t+t^{-1},\quad y=1+t^2$.
(vi) $x=t^2/(1+t),\quad y=t^3/(1+t)$.
(vii) $x=(t+1)^6/t^4,\quad y=(t+1)^4/t^3$.
(viii) $x=t(2-t),\quad y=t^4(2-t)^1$.
(ix) $x=(1+t+t^2)/(t+t^2),\quad y=(1+t+t^2)/(1+t)$.

[(i) Approximations at the origin $y^2=3ax$ and $x^2=3ay$; at infinity $x+y+a=0$. Symmetrical about $x=y$, as is seen by changing t into $1/t$.

(ii) Approximations at origin $y^3 = 5ax^2$ and $x^3 = 5ay^2$; at infinity $x+y = a$. Symmetrical about $x = y$.

(iii) Approximation at infinity $y^4 = x^3$. Tangents parallel to axes of x and y where $t = 0$ (inflexion) and $t = -2^{\frac{1}{3}}$.

(iv) Approximation at origin $2y^2 = -9x$; at infinity $y^4 = x^3$. Symmetrical about $y = 0$. Node at $t = \pm\sqrt{3}$, cusps at $t = \pm 1$.

(v) Approximations at infinity $y = 1$ and $x^2 = y$. Symmetrical about $x = 0$.

(vi) Approximation at origin $y^2 = x^3$; at infinity $x+y = 1$ and $x^2 = y$.

(vii) At origin a rhamphoid cusp. At infinity approximates to $x^3 = y^4$ and $x = y^2$.

(viii) Approximations at origin $x^4 = 2y$ and $2x^3 = y$; at infinity $y^2 = -x^7$.

(ix) Acnode at origin. Asymptotes $x = 1$, $y = 1$, $x+y+1 = 0$. Symmetrical about $x = y$.]

Ex. 3. Trace the curves

(i) $x = a\cos^3 t, \quad y = a\sin^3 t.$
(ii) $x = a\cos^6 t, \quad y = a\sin^6 t.$
(iii) $x = a\sin 3t, \quad y = a\cos t.$
(iv) $x = a\cos^2 t \,.\, \cos 2t, \quad y = a\sin^2 t \,.\, \sin 2t.$
(v) $x = a\cosh^3 t, \quad y = b\sinh^3 t.$
(vi) $x = a\cos 3t \,.\, \operatorname{cosec} 2t, \quad y = a\sin 3t \,.\, \operatorname{cosec} 2t.$

§ 10.

Some curves are most readily traced by turning the Cartesian into the polar equation.

For instance, the curve

$$(x^2+y^2-bx)^2 = a^2(x^2+y^2)$$

becomes in polar coordinates

$$r = a+b\cos\theta.$$

This is readily drawn by noting how r alters as θ increased from 0 to 2π.* The case of $5a = 6b$ is shown in Ch. IV, Fig. 1.

The asymptotes are given by noting that the polar subtangent $r^2\dfrac{d\theta}{dr}$ becomes the perpendicular from the pole on an asymptote, if θ has a value α which makes r infinite. The perpendicular is positive or negative according as the asymptote lies on the right or left of a person standing at the pole and looking along the direction $\theta = \alpha$.

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* It may also be drawn by increasing each radius vector of the circ' $r = b\cos\theta$ by the constant a.

Ex. 1. Trace the curves

(i) $(x^2+y^2)^2 = a^2(x^2-y^2)$.

(ii) $(x^2+y^2)(x-b)^2 = a^2x^2$.

(iii) $(x^2+y^2)^2 = ay(3x^2-y^2)$.

(iv) $(x^2+y^2)^3 = a^2y^4$.

(v) $x^2(x^2+y^2) = a^2y^2$.

(vi) $(x^2+y^2-2a^2)^2 = x^2(x^2+y^2)$.

(vii) $y^2(x^2+y^2-a^2)^2 = 4a^2x^2(x^2+y^2)$.

(viii) $(4x^2+y^2)x^2y^2 = a^2(x^2-y^2)^2$.

Ex. 2. Trace the curve

$$(x^2+4y^2)^5 = a^2(x^4-24x^2y^2+16y^4)^2.$$

[The locus of the mid-points of the ordinates of

$$(x^2+y^2)^5 = a^2(x^4-6x^2y^2+y^4)^2.]$$

CHAPTER IV

TANGENTIAL EQUATION AND POLAR RECIPROCATION

§ 1. Tangential Equation.

IF the line

$$\lambda x+\mu y+\nu z=0$$

touches a given curve whose equation in homogeneous coordinates is $f(x, y, z)=0$, a certain relation, say $\phi(\lambda, \mu, \nu)=0$, must hold between λ, μ, ν. This relation is called the *tangential equation* of the curve.

It is homogeneous in λ, μ, ν, for it is not altered if we multiply λ, μ, ν by the same constant.

We may call $f(x, y, z)=0$ the *point-equation* of the curve, if we wish to emphasize the difference between it and the tangential equation.

If the equation of the curve is given in Cartesian coordinates, the tangential equation is the condition that the curve should touch

$$\lambda x+\mu y+1=0.$$

It is not homogeneous in λ, μ in general.

Suppose that the triangle of reference of homogeneous coordinates is altered. We shall obtain the new equation of a curve $f(x, y, z)=0$ by putting in this equation

$$x=l_1 x'+m_1 y'+n_1 z', \qquad y=l_2 x'+m_2 y'+n_2 z',$$
$$z=l_3 x'+m_3 y'+n_3 z' \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{i})$$

$l_1, m_1, \ldots, n_3$ being constants; and then dropping the dashes.

We get the new tangential equation of the curve by putting in the original tangential equation $\phi(\lambda, \mu, \nu)=0$,

$$\lambda=L_1 \lambda'+L_2 \mu'+L_3 \nu', \qquad \mu=M_1 \lambda'+M_2 \mu'+M_3 \nu',$$
$$\nu=N_1 \lambda'+N_2 \mu'+N_3 \nu' \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{ii})$$

and then dropping the dashes; where equations (i) give on solving for x, y, z

$$x'=L_1 x+M_1 y+N_1 z, \qquad y'=L_2 x+M_2 y+N_2 z,$$
$$z'=L_3 x+M_3 y+N_3 z.$$

For on changing the homogeneous coordinates by the substitution given by equations (i), the line $\lambda x+\mu y+\nu z=0$ becomes

$$(l_1\lambda+l_2\mu+l_3\nu)x'+(m_1\lambda+m_2\mu+m_3\nu)y' \\ +(n_1\lambda+n_2\mu+n_3\nu)z'=0.$$

This is identical with $\lambda'x'+\mu'y'+\nu'z'=0$, if

$$\lambda'=l_1\lambda+l_2\mu+l_3\nu, \quad \mu'=m_1\lambda+m_2\mu+m_3\nu, \\ \nu'=n_1\lambda+n_2\mu+n_3\nu \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}.$$

But equations (iii) give equations (ii) on solving for λ, μ, ν; which proves the result.

§ 2. Class of a Curve.

A line is considered as 'touching' a point in the sense of § 1 if it passes through it. The line $\lambda x+\mu y+\nu z=0$ passes through the point (x', y', z') if

$$\lambda x'+\mu y'+\nu z'=0.$$

This is therefore the tangential equation of the point. It is to be noted that the tangential equation of a point is of the first degree. A straight line has no tangential equation.

If the tangential equation $\phi(\lambda, \mu, \nu)=0$ of a curve is homogeneous of degree m in λ, μ, ν, the tangents from (x', y', z') to the curve are given by solving for $\lambda:\mu:\nu$ from

$$\lambda x'+\mu y'+\nu z'=0, \quad \phi(\lambda, \mu, \nu)=0.$$

Eliminating ν from these two equations, we have an equation of degree m in $\lambda:\mu$. Hence m tangents can be drawn from any point to the curve. This number is called the *class* of the curve. As a particular case, m tangents can be drawn to the curve in any given direction.

The common tangents of two curves of class m and M,

$$\phi(\lambda, \mu, \nu)=0 \text{ and } \Phi(\lambda, \mu, \nu)=0,$$

are found by solving these two equations for $\lambda:\mu:\nu$. They are mM in number. The proof is exactly similar to that used in Ch. I, § 7.

The point of contact of any tangent having the point-equation

$$\lambda'x+\mu'y+\nu'z=0$$

with the curve whose tangential equation is $\phi(\lambda, \mu, \nu)=0$ has the tangential equation

$$\lambda\frac{\partial f}{\partial\lambda'}+\mu\frac{\partial f}{\partial\mu'}+\nu\frac{\partial f}{\partial\nu'}=0;$$

where $\frac{\partial f}{\partial \lambda'}$ means the result of putting λ', μ', ν' for λ, μ, ν in $\frac{\partial f}{\partial \lambda}$, &c. The proof is similar to that of Ch. I, § 9.

As in Ch. I, § 8, we may call $u+kv=0$ a (tangential) *pencil* of curves of the m-th class, u and v being homogeneous of degree m in λ, μ, ν.

The curves obtained by taking different values of k have m^2 common tangents, namely the common tangents of $u=0$ and $v=0$.

§ 3. Tangential Equation of any Curve.

To obtain the tangential equation of

$$f(x, y, z) = 0,$$

we find the lines

$$f(x, y, -(\lambda x+\mu y)/\nu) = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

joining its intersections with the line

$$\lambda x+\mu y+\nu z = 0$$

to the vertex $(0, 0, 1)$ of the triangle of reference. Two of the lines represented by (i) are coincident if $\lambda x+\mu y+\nu z=0$ touches the curve. Hence, if we find the condition that (i) considered as an equation in x/y has equal roots, by eliminating x/y between (i) and the result of differentiating (i) with respect to x/y, we get the required tangential equation.

Similarly, to obtain the point-equation when the tangential equation $\phi(\lambda, \mu, \nu)=0$ is given, we find the condition that two of the intersections of the curve with a side of the triangle of reference

$$\phi(\lambda, \mu, -(\lambda x+\mu y)/z) = 0$$

should coincide, i. e. that this last equation should have equal roots considered as an equation in λ/μ.

An alternative method of finding the tangential equation of $f(x, y, z)=0$ is as follows.

Suppose (x', y', z') is the point of contact of the tangent $\lambda x+\mu y+\nu z=0$. This line must pass through (x', y', z') and be identical with the tangent

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}=0$$

to $f=0$ at (x', y', z').

We therefore obtain the required tangential equation by eliminating (x', y', z') between

$$\lambda x'+\mu y'+\nu z'=0 \quad \text{and} \quad \frac{\partial f}{\partial x'}/\lambda = \frac{\partial f}{\partial y'}/\mu = \frac{\partial f}{\partial z'}/\nu.$$

Again, if the coordinates of any point of the curve are known functions of a parameter t, the equation of the tangent at the point is $ax+by+cz=0$, where a, b, c are known functions of t. (See Ch. I, § 9.)

Eliminating t between $a/\lambda = b/\mu = c/\nu$, we have the tangential equation required.*

Ex. 1. The tangential equation of

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0,$$

where $A=bc-f^2$, $F=gh-af$, &c.

Ex. 2 A conic is of degree 2 and class 2.

Ex. 3. If in Ex. 1 the point-equation is deduced from the tangential in the same way as the tangential from the point-equation, we have the original equation multiplied by $(abc+2fgh-af^2-bg^2-ch^2)$.

Ex. 4. If in Ex. 1 the original equation is a line-pair, the tangential equation is the intersection of the lines twice over.

If the original equation is a pair of coincident lines, the tangential equation vanishes identically.

Ex. 5. Find the tangential equations of

(i) $kx^3 = y^2z$.
(ii) $ay^2z = x(x^2+y^2)$.
(iii) $3(x+y) = x^3$.
(iv) $x(x^2+y^2) = ay^2$.
(v) $a^3z^3 = xy(x+y+az)$.
(vi) $(x^2+y^2)^2 = a^2(x^2-y^2)$.
(vii) $x^{\frac{2}{3}}+y^{\frac{2}{3}} = a^{\frac{2}{3}}z^{\frac{2}{3}}$.
(viii) $y^pz^q = x^{p+q}$.

[(i) $27k\mu^2\nu+4\lambda^3=0$.
(ii) $27a^2\mu^2\nu+4(a\lambda+\nu)^3=0$.
(iii) $9\mu+4(\lambda-\mu)^3=0$.
(iv) $27a^2\mu^2+4(a\lambda+1)^3=0$.
(v) $27a^4\lambda^2\mu^2-18a^2\lambda\mu\nu+4a(\lambda+\mu)\nu^2-\nu^3=0$.
(vi) $27a^4(\lambda^2+\mu^2)^2=(4-a^2\lambda^2+a^2\mu^2)^3$.
(vii) $(\lambda^2+\mu^2)\nu^2=a^2\lambda^2\mu^2$.
(viii) $(-\lambda)^{p+q}p^pq^q=\mu^p\nu^q(p+q)^{p+q}$.]

Ex. 6. The product of the perpendiculars from m fixed points on a line is proportional to the cosine of m times the angle which these perpendiculars make with a fixed direction. Show that the envelope of the line is a curve of class m, and that the tangents from any one of the fixed points are parallel to the sides of the same regular polygon.

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* The third method is usually best when it can be applied. The second method is best in Ex. 1.

Ex. 7. The envelope of the asymptotes of the pencil of n-ics $S+kS'=0$ is of class $2n-1$.

Any one of these asymptotes meets $S=0$ and $S'=0$ in two sets of n points with the same centroid.

The envelope touches the line at infinity at each of the $2(n-1)$ points where a curve of the pencil touches it.

[It follows from Ch. VIII, § 1, that the degree of the envelope is $4(n-1)$ in general.]

§ 4. Tangential Equation of the Circular Points.

The tangential equation of the circular points is obtained by writing down the condition that any line should be perpendicular to itself. For instance, in rectangular Cartesian coordinates the tangential equation of the circular points is

$$\lambda^2+\mu^2=0,$$

for this is the condition that $\lambda x+\mu y+1=0$ should be self-perpendicular.

Ex. 1. The circular points are

$$\lambda^2-2\lambda\mu\cos\omega+\mu^2=0$$

for Cartesian axes inclined at an angle ω.

Ex. 2. The circular points are

$$\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\nu\lambda\cos B-2\lambda\mu\cos C=0$$

for trilinear coordinates.

Ex. 3. Write down the tangential equation in trilinear coordinates of a conic with foci $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ and minor axis $2k$.

Deduce the equation of a circle with given centre and radius, e.g. the circle inscribed in the triangle of reference.

$$[(\lambda\alpha_1+\mu\beta_1+\nu\gamma_1)(\lambda\alpha_2+\mu\beta_2+\nu\gamma_2)$$
$$=k^2(\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\nu\lambda\cos B-2\lambda\mu\cos C).]$$

Ex. 4. Obtain the equation of the director-circle of a conic whose equation is given in any coordinates.

[The locus of a point such that the tangents from it to the conics

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu=0,$$
$$A'\lambda^2+B'\mu^2+C'\nu^2+2F'\mu\nu+2G'\nu\lambda+2H'\mu\nu=0$$

form an harmonic pencil is

$$(BC'+B'C-2FF')x^2+\ldots+\ldots$$
$$+2(GH'+G'H-AF'-A'F)yz+\ldots+\ldots=0.$$

Take the second conic as the circular points, and this 'harmonic locus' becomes the director-circle of the first conic.]

Ex. 5. A curve of class m touches the line at infinity at the circular points and $m-3$ other points. Show that the sum of the angles which the tangents from any point make with a fixed line is constant to within a multiple of π.

[The curve is $f(\lambda, \mu)+(\lambda^2+\mu^2)\phi(\lambda, \mu)=0$; f and ϕ being homogeneous of degrees m and $m-3$ respectively in λ and μ. The inclination of the tangents from (x, y) to $y=0$ is given by $f=(\lambda^2+\mu^2)(\lambda x+\mu y)\phi$.]

Ex. 6. More generally, the sum of the angles which the common tangents to this curve and any curve of given class make with a fixed line is constant to within a multiple of π.

Discuss the case $m = 3$.

[(i) As in Ex. 5. (ii) The three-cusped hypocycloid.]

§ 5. Polar Reciprocation.

Suppose we have a certain 'base-conic'. The envelope of the polar of any point on a given curve Σ is called the *polar reciprocal* Σ' of Σ with respect to the base-conic.

Consider two consecutive points P and Q on Σ. Their polars meet at the pole R of PQ with respect to the base-conic. But these two polars are consecutive tangents to Σ' in the limit, and R is the point of contact of either. Also PQ is in the limit the tangent at P to Σ. Hence Σ is the envelope of the polars of points on Σ'; or the relation between the curves is a reciprocal one, a point on Σ and the tangent at this point being the pole and polar with respect to the base-conic of a corresponding tangent to Σ' and its point of contact.

If l is any line, its intersections with Σ are the poles of the tangents to Σ' from the pole of l with respect to the base-conic. Hence:

The degree of a curve is equal to the class of its polar reciprocal.

§ 6. Equation of Polar Reciprocal.

The polar reciprocal of the curve having tangential equation $\phi(\lambda, \mu, \nu) = 0$ *with respect to the base-conic* $x^2+y^2+z^2 = 0$ *is* $\phi(x, y, z) = 0$.

For the polar of (x', y', z') with respect to the base-conic is

$$xx'+yy'+zz' = 0;$$

and this touches $\phi(\lambda, \mu, \nu) = 0$, if $\phi(x', y', z') = 0$. But (x', y', z') lies on the polar reciprocal in this case.

Hence the polar reciprocal of a curve can be obtained when its tangential equation is known, which is always the case when its point-equation is given.

The reader will notice the close connexion which exists between the algebraic notion of 'tangential equation' and the geometrical conception of 'polar reciprocal'.

Ex. 1. If the condition that a curve (in Cartesian coordinates) should touch $\lambda x+y+\nu = 0$ is $\phi(\lambda, \nu) = 0$, the polar reciprocal with respect to $x^2+2y = 0$ is $\phi(x, y) = 0$.

[The polar of (x', y') with respect to $x^2+2y = 0$ is $xx'+y+y' = 0$.]

Ex. 2. The polar reciprocal of a 'Lamé Curve' $(x/a)^n+(y/b)^n = 1$ with respect to $(x/A)^2+(y/B)^2 = 1$ is a Lamé Curve.

Ex. 3. The base-conic, any conic, and its polar reciprocal have in general a common self-conjugate triangle.

Ex. 4. The polar reciprocal of $\phi(\lambda, \mu, \nu) = 0$ with respect to

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is

$$\phi(ax + hy + gz,\ hx + by + fz,\ gx + fy + cz) = 0.$$

§ 7. Singularities of Curve and its Reciprocal.

We shall now consider what corresponds in the polar reciprocal to the 'singularities' of a curve, i. e. its nodes, cusps, &c.

A node is a point of a curve at which there are two tangents.

Hence to a node corresponds in the reciprocal curve a tangent with two points of contact. Such a tangent is called a *bitangent.*

If the base-conic has a real equation, to a crunode corresponds a real bitangent with two real points of contact, and to an acnode corresponds a real bitangent with unreal points of contact.*

In the same way, to a triple, quadruple, . . . point with distinct tangents corresponds a tangent with three, four, . . . distinct points of contact, which may be called a triple, quadruple, . . . tangent.

To a cusp C of a curve and its cuspidal tangent i correspond an inflexional tangent c of the reciprocal curve and its inflexion I.

For the cusp has the properties that every line through C meets the curve twice at C, except i, which meets it thrice at C; while of the tangents from any point P on i one coincides with i, unless P is at C, when three of the tangents coincide with i.†

Hence from any point on c two tangents can be drawn to the reciprocal curve coinciding with c unless the point is at I, when three of the tangents coincide with c; while any line through I meets the reciprocal curve once at I, except c, which meets it thrice at I. Therefore I is an inflexion and c the inflexional tangent.

Hence:

To a node of a curve and its tangents correspond a bitangent of the reciprocal curve and its points of contact, to a cusp and its cuspidal tangent correspond an inflexional tangent and

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* Sometimes called an 'ideal' bitangent. For pictures of bitangents see Ch. IV, Fig. 1; Ch. X, Fig. 1; Ch. XVIII, Figs. 2, 6, 7, 9, 10.

† These properties of the cusp are evident from the fact that, by Newton's diagram, any curve approximates near a cusp to a semi-cubical parabola $ay^2 = x^3$, for which the properties are at once established. They are almost intuitive from a figure. Similarly for the inflexion.

A more rigorous proof is given in Ch. VI, §§ 3, 5.

its inflexion. More generally, to a k-ple point and its k tangents correspond a k-ple tangent and its k points of contact.

The relations between the singularities of a curve and its reciprocal are illustrated by Fig. 1. This diagram shows the limaçon $r = 6+5\cos\theta$ and its reciprocal with respect to the circle $4(r^2-3r\cos\theta) = 55$.

The singularities of the limaçon are an acnode, two unreal cusps (at the circular points), a bitangent with real points of contact, and two real inflexions. These reciprocate respectively into an ideal bitangent, two unreal inflexional tangents, a crunode, and two real cuspidal tangents. The degree and class of the limaçon and its reciprocal are both four. The two bitangents are shown by the dotted lines.

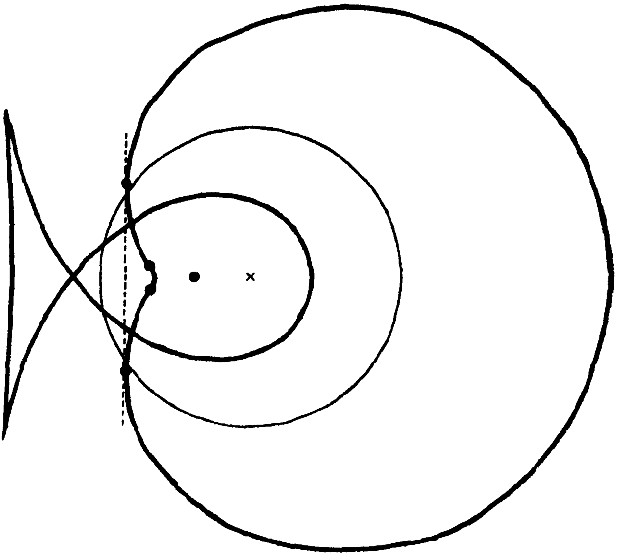

Fig. 1.

Ex. 1. To a triple point with three coincident tangents corresponds in general the tangent at a point of undulation ; and so for quadruple, quintuple, ... points.

Ex. 2. If O is a multiple point of a curve with distinct tangents, two of the tangents from O to the curve coincide with each tangent at O.

Ex. 3. If $\phi(\lambda, \mu, \nu) = 0$ is the tangential equation of a curve, the bitangents are given by
$$\frac{\partial\phi}{\partial\lambda} = \frac{\partial\phi}{\partial\mu} = \frac{\partial\phi}{\partial\nu} = 0.$$
Find the inflexions.
[Compare Ch. II, § 4.]

Ex. 4. The singularities of $y^p z^q = x^{p+q}$ at (0, 1, 0) and (0, 0, 1) are reciprocals of each other.

[Use § 3, Ex. 5 (viii).]

Ex. 5. A quintic cannot have a triple point and three cusps.

[Its reciprocal would be a quintic with a triple tangent.
Note that to be given that a curve has a triple point and three cusps is equivalent to only $4+6$ conditions, whereas a quintic can satisfy 20 conditions in general.]

Ex. 5. A sextic cannot have a triple point, a node, and six cusps.

Ex. 7. A 7-ic cannot have a quadruple point, four nodes, and five cusps.

§ 8.

If a curve has degree n, class m, δ nodes, κ cusps, τ bitangents, and ι inflexions, we have shown, in § 7, that the polar reciprocal has degree m, class n, τ nodes, ι cusps, δ bitangents, and κ inflexions.

If the point-equation of a curve is written down at random, it has no node or cusp (Ch. II, § 6), but we shall see that it has bitangents and inflexions. The point-equation of its reciprocal must not be considered as 'written down at random', for this reciprocal curve is specialized by the fact that it has been derived by reciprocation from a curve with random point-equation. In fact the reciprocal has both nodes and cusps in general.

If the tangential equation of a curve is written down at random, the curve will have no bitangent or inflexion, but will have nodes and cusps. In fact the writing down of the tangential equation of a curve is equivalent to writing down the point-equation of its polar reciprocal with respect to the base-conic $x^2+y^2+z^2=0$.

Suppose a curve is subjected to r conditions; then its polar reciprocal with respect to any given conic is also subjected to r conditions. For instance, if the given curve is made to pass through r assigned points, the reciprocal has r assigned tangents, &c.

Suppose we are told that the curve is of degree n, has δ nodes, has κ cusps, and satisfies r other conditions; and also that only a finite number of curves can be found with these properties. Then (Ch. II, § 6)

$$\tfrac{1}{2}n(n+3) = \delta + 2\kappa + r.$$

The polar reciprocal is of degree m, has τ nodes, ι cusps, and satisfies r other conditions, and only a finite number of polar reciprocals exist; the base-conic being supposed given throughout. Hence

$$\tfrac{1}{2}m(m+3) = \tau + 2\iota + r.$$

We deduce

$$\tfrac{1}{2}n(n+3) - \delta - 2\kappa = \tfrac{1}{2}m(m+3) - \tau - 2\iota.$$

Ex. 1. A singly infinite family of curves has the equation

$$f(x, y, z, a) = 0,$$

which is an algebraic equation of degree p in the parameter a. Show that the tangential equation of the family $\phi(\lambda, \mu, \nu, a) = 0$ is also an algebraic equation in a, say of degree l.

Show that the general p curves of the family pass through any given point, and l touch any given line.

[(p, l) is called the *characteristic* of the family.]

Ex. 2. The characteristic of the polar reciprocal of the family is (l, p).

Ex. 3. Find the characteristics of the following families:

(i) Conics through r points and touching s lines, where $r+s = 4$.

(ii) Conics touching two given lines at given points.

(iii) Conics through two given points touching a given line at a given point.

(iv) Conics through a given point, touching a given line, and touching a given line at a given point.

(v) Circles touching a given circle with their centres on a given line.

(vi) Conics with a given focus, point, and length of major axis.

(vii) Conics with given vertices.

(viii) Conics with axes along given lines and passing through a given point.

(ix) Conics with given centre and eccentricity, passing through a given point.

(x) The circles of curvature of a given parabola.

[(i) $(1, 2)$, $(2, 4)$, $(4, 4)$, $(4, 2)$, $(2, 1)$ as $r = 4, 3, 2, 1, 0$. (ii) $(1, 1)$. (iii) $(1, 2)$. (iv) $(2, 2)$. (v) $(2, 4)$. (vi) $(10, 6)$. (vii) $(1, 1)$. (viii) $(1, 2)$. (ix) $(2, 4)$. (x) $(4, 6)$.]

Ex. 4. Find the characteristics of

(i) A pencil of n-ics.

(ii) Curves of degree n and class m with a common centre of similitude.

(iii) The family obtained by rotating a curve through any angle about a fixed point.

(iv) The family obtained by translating a curve through any distance in a given direction.

(v) Curves parallel to a given curve.

[(i) $(1, 2n-2)$. (ii) (n, m). (iii) $(2n, 2m)$. (iv) (n, m). (v) $(n+m, m)$. See Ch. XI, § 2, Ex. 4, and § 10.]

Ex. 5.* Find the characteristics of

(i) Cubics with a given cusp, cuspidal tangent, inflexion, and inflexional tangent.

(ii) Cubics with a given cusp, point, inflexion, and inflexional tangent.

(iii) Cubics with a given cusp, inflexion, tangent, and point of contact.

[(1, 1), (2, 3), (2, 3). The families are
$zy^2 = ax^3, \quad z(y+ax)^2 = 2axy^2 + a^2x^2y, \quad z(y+ax)^2 = axy^2 + 2a^2x^2y.$]

Ex. 6. Find the characteristics of

(i) Cubics with a given node, nodal tangents, and inflexions.

(ii) Nodal cubics with three given collinear inflexions, and given tangents at two of them.

[(1, 4) and (3, 4). The families are

$$ax(z^2+y^2) = y(3z^2-y^2), \qquad a^2z^3 = xy(x+y+3az).]$$

Ex. 7. Find the characteristics of

(i) Cubics with nine given inflexions.

(ii) Cubics with three given collinear inflexions and corresponding inflexional tangents.

[(1, 4) and (1, 2). See Ch. XIV, §§ 5 and 8.]

Ex. 8. Find the characteristics of

(i) Lemniscates of Bernouilli with a given node and axis.

(ii) Quartics with a given node, two given cusps, and given tangents at these cusps.

(iii) Quartics with three given biflecnodes and given tangents at one of the biflecnodes.

(iv) Quartics with three given nodes and four other given points.

[(1, 3), (1, 3), (1, 3), (1, 6). See § 3, Ex. 5 (vi), and Ch. XVII, §§ 3, 6.]

Ex. 9. Find the characteristic of the conics having 5-point contact with a given cuspidal cubic.

[(6, 6). See Ch. XIII, § 3, Ex. 7.]

Ex. 10. The complete primitive of the differential equation

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0,$$

which is algebraic and of degree μ in $\frac{dy}{dx}$, while $\phi(x, mx, m) = 0$ is an equation of degree ν in x, is a family with characteristic (μ, ν).

Ex. 11. Find the Plücker's numbers of the locus of the point of contact of tangents from a given point O to a family of curves with characteristic (p, l).

[$n' = p + l$, $m' = 2p + M$, $\kappa' = 0$ in general, where M is the class of the envelope of the family. O is a p-ple point of the locus. Consider the intersections with the locus of any line through O, and the tangents from O to the locus. Verify by considering conics through four points or touching four lines.]

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* Ex. 5 to 13 should be omitted on a first reading.

Ex. 12. Find the degree and class of the locus of the intersection of tangents from fixed points A and B to a family of curves of degree n and class m with characteristic (p, l).

[$n' = l(2m-1)$, $m' = m(M+p)+2l(m-1)$, where M is the class of the envelope. A and B are $l(m-1)$-ple points of the locus and the tangents from A and B to the envelope are m-ple tangents of the locus.]

Ex. 13. Find the degree and class of the locus of the foci of a family of curves of degree n and class m with characteristic (p, l). The foci of the envelope are m-ple foci of the locus.

[In Ex. 12 take A and B at the circular points.]

CHAPTER V

FOCI

§1. Definition of Foci.

If ω and ω' are the circular points, and the lines $S\omega$, $S\omega'$ touch a given curve (not at ω and ω'), S is called a *focus* of the curve.

If the curve is of class m, m tangents can in general be drawn to the curve from ω and m from ω'. If such a tangent is $x+iy=a+ib$, then $x-iy=a-ib$ is also a tangent, the curve being supposed real.

These tangents meet in the *real* focus (a, b), so that there are m real foci. There are no more than m real foci, for no tangent from ω can contain more than one real focus, since the line joining two real points is real, and cannot pass through ω or ω'.

Hence:

A curve of class m has in general m real and m^2-m unreal foci.

If $f(\lambda, \mu, \nu)=0$ is the tangential equation of the curve, any curve with the same foci is

$$f(\lambda, \mu, \nu)+\phi(\lambda, \mu, \nu)\cdot\psi(\lambda, \mu, \nu)=0 \quad . \quad . \quad . \quad \text{(i)},$$

where $\phi(\lambda, \mu, \nu)=0$ is the tangential equation of the circular points, and $\psi(\lambda, \mu, \nu)=0$ is any curve of class $m-2$.

For to say that a curve is confocal with $f=0$ is equivalent to saying that it touches the $2m$ common tangents of $f=0$ and $\phi=0$. Hence the general tangential equation of a curve confocal with $f=0$ has

$$\tfrac{1}{2}m(m+3)-2m=\tfrac{1}{2}m(m-1)$$

arbitrary coefficients. But (i) evidently touches the common tangents of $f=0$ and $\phi=0$ and has $\frac{1}{2}m(m-1)$ arbitrary coefficients, since this is the number of coefficients in ψ.

As a special case, the general tangential equation of the curve whose m real foci have tangential equations $f_1=0$, $f_2=0, \dots, f_m=0$ is

$$f_1 f_2 \cdots f_m+\phi(\lambda, \mu, \nu)\cdot\psi(\lambda, \mu, \nu)=0,$$

where $\phi = 0$ is the circular points and $\psi = 0$ any curve of class $m-2$.

Not many properties of the foci of the general curve are known (but see the examples below). The foci of special curves may have interesting geometrical properties.

That the 'foci' as defined in this section are the 'foci' of the conic, when $m = 2$, according to the usual definition of foci of a conic may be shown as follows.

If $P(x, y)$ is any point of a conic, $S(x_1, y_1)$ a focus, e the eccentricity, and $p = x \cos\alpha + y \sin\alpha$ the directrix, the equation of the conic is

$$(x-x_1)^2 + (y-y_1)^2 = e^2(p - x\cos\alpha - y\sin\alpha)^2;$$

since $SP = e \, . \, PM$, where PM is the perpendicular from P to the directrix.

It is seen at once that the circular lines through S, namely

$$(x-x_1)^2 + (y-y_1)^2 = 0$$

touch the conic, and that the directrix is the chord of contact.

By analogy we may call the chord of contact of the two circular lines through a focus S of any curve the *directrix* corresponding to S.

Ex. 1. If the middle point P of the line joining two foci of a real curve is real, P is the middle point of the line joining two *real* foci.

Ex. 2. The coaxial family of circles through two foci of a curve have two other foci as limiting points.

Ex. 3. Tangents are drawn from any point O to two confocal curves. The tangents to one curve make with any fixed line angles whose sum is α, and the tangents to the other curve make angles whose sum is β. Show that $\alpha - \beta$ is a multiple of π.

[Taking rectangular Cartesian axes through O and the fixed line parallel to the axis of x, suppose the terms of highest degree in the tangential equations of the curves to be

$$p_0\lambda^m - p_1\lambda^{m-1}\mu + p_2\lambda^{m-2}\mu^2 - \ldots \quad \text{and} \quad q_0\lambda^m - q_1\lambda^{m-1}\mu + q_2\lambda^{m-2}\mu^2 - \ldots.$$

Since the difference of these has $\lambda^2 + \mu^2$ as a factor.

$$\tan\alpha = (-p_1 + p_3 - p_5 + \ldots) \div (p_0 - p_2 + p_4 - \ldots)$$

and

$$\tan\beta = (-q_1 + q_3 - q_5 + \ldots) \div (q_0 - q_2 + q_4 - \ldots)$$

are equal.]

Ex. 4. The tangents from O to a curve make with a fixed line angles whose sum is α, and the lines joining O to the foci make angles whose sum is β. Show that $\alpha - \beta$ is a multiple of π.

[A particular case of Ex. 3. The theorem is well known in the case of the conic.]

Ex. 5. The sum of the angles (to within a multiple of π) which the common tangents to two curves c and c' make with any fixed line is not altered, if c and c' are replaced by curves respectively confocal with them.

[See *American Journal Math.*, x, p. 58, and xii, p. 161.]

Ex. 6. If the two curves of the family $Pa^2+Qa+R=0$ which pass through any point are always orthogonal, where P, Q, R are polynomials in x, y and a is a parameter, then the curves of the family are confocal.

[The envelope of the family must evidently consist of circular lines. The case in which the family consists of conics is familiar.]

Ex. 7. If four foci are concyclic, the curve has three other sets of four concyclic foci. The four circles cut orthogonally.

[If a, b, c, d are tangents from ω and a', b', c', d' tangents from ω' such that the intersections of aa', bb', cc', dd' are concyclic, the pencils $(abcd)$, $(a'b'c'd')$ have the same cross-ratio. Therefore so do the pencils $(abcd)$, $(b'a'd'c')$, &c.]

§ 2. Singular Foci.

If a curve of class m touches the line at infinity $\omega\omega'$, the number of tangents, other than $\omega\omega'$, which can be drawn to the curve from ω and ω', and by their intersections determine the foci, is less than in the general case; so that the number of foci is less than m^2.*

For instance, if the curve has ordinary contact at one point with $\omega\omega'$, it will have $(m-1)^2$ foci, of which $m-1$ are real. A well-known example is the parabola.

If the curve passes through ω and ω', the tangents at ω and ω'† meet in points which are not usually included among the ordinary 'foci'. They will be called *singular foci*. If the curve has k-ple points at ω and ω', it will have in general k^2 singular foci of which k are real, and $(m-2k)^2$ ordinary foci of which $m-2k$ are real (see Ch. IV, § 7, Ex. 2).

For example, the centre of a circle is a singular focus, and the circle has no ordinary foci.

Ex. 1. The projection of a curve nearly touches the projection of $\omega\omega'$. Show that the projection of $(m-1)$ foci are close to the projection of ω, and so for ω'; and that the projection of another focus is near the point of approach of the projection of $\omega\omega'$ and the curve.

Ex. 2. The projection of a curve nearly passes through the projections of ω and ω'. Show that the projections of four of the foci are close together and in the limit coalesce at the projection of a singular focus.

Ex. 3. Find a curve with no focus, singular or ordinary.

[A curve of class m touching $\omega\omega'$ at ω, ω' and $m-3$ other points; or touching $\omega\omega'$ at ω, ω' and $m-4$ other points, and having cusps at ω and ω'.]

Ex. 4. The envelope of a line of given length with its ends one on each of two given curves has no (ordinary) foci.

[A segment through ω must have zero length.]

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* A point is not called a 'focus' unless it is finite.

† Excluding $\omega\omega'$, if this is a tangent. The intersection of a tangent *at* ω with a tangent *from* ω', or vice versa, is not counted as either a singular or an ordinary focus.

Ex. 5. If $\Sigma = 0$, $\Sigma' = 0$ are the tangential equations of two curves of the m-th class, the locus of the foci of the pencil $\Sigma + k\Sigma' = 0$ is a $(2m-1)$-ic having an $(m-1)$-ple point at ω and ω', and passing through its own singular foci.

[By § 1 if (x_r, y_r) and (x'_r, y'_r) are the foci of

$$\Sigma = 0 \text{ and } \Sigma' = 0 \quad (r = 1, 2, \ldots, m),$$

$$\Sigma \equiv \prod_r (\lambda x_r + \mu y_r + 1) + (\lambda^2 + \mu^2)\,\psi, \text{ and so for } \Sigma'.$$

Writing down the conditions that the lines joining (x, y) to ω and ω should touch $\Sigma + k\Sigma' = 0$, and eliminating k, we have for the locus

$$\Pi (x_r + iy_r - x - iy)(x'_r - iy'_r - x + iy)$$
$$= \Pi (x_r - iy_r - x + iy)(x'_r + iy'_r - x - iy).$$

For another proof see Ch. XXI, § 2, Ex. 4, and Ch. IV, § 8, Ex. 13]

Ex. 6. The lines joining any point of the locus of Ex. 5 to the foci of $\Sigma = 0$ and $\Sigma' = 0$ make angles with any fixed line whose sum is the same for both curves to within a multiple of π.

[If α is the angle which the line joining (x, y) and (x_r, y_r) makes with $y = 0$, $\quad e^{2i\alpha} = (x_r + iy_r - x - iy)/(x_r - iy_r - x + iy)$.]

Ex. 7. The tangents from the real foci of $\Sigma' = 0$ to any curve of the tangential pencil $\Sigma + k\Sigma' = 0$ touch a curve confocal with $\Sigma = 0$.

Ex. 8. The sum of the angles made with a fixed line by the tangents to a curve of class m from any point of an m-ic through all its foci is constant.

The asymptotes of the latter curve are parallel to the sides of a regular polygon.

[See Ex. 6, and § 1, Ex. 4.]

Ex. 9. If $f(x) = 0$ is an algebraic equation of degree n with roots $\alpha_1 + i\beta_1$, $\alpha_2 + i\beta_2$, ..., and the roots of $f'(x) = 0$ are $\alpha'_1 + i\beta'_1$, $\alpha'_2 + i\beta'_2$, ..., prove that the centroid of the points (α_1, β_1), (α_2, β_2), ... coincides with the centroid of the points (α'_1, β'_1), (α'_2, β'_2), ...

Show that there exists a curve of class $n-1$ with real foci at the former points which touches at its middle point the line joining any two of the latter points.

[The curve is $(\lambda\alpha_1 + \mu\beta_1 + 1)^{-1} + (\lambda\alpha_2 + \mu\beta_2 + 1)^{-1} + \ldots = 0$.]

Ex. 10. Fixed segments A_1B_1, A_2B_2, ..., A_mB_m subtend angles at P whose sum σ is constant. Show that the locus of P is a m-circular $2m$-ic.

If σ is an odd multiple of $\frac{1}{2}\pi$, the real singular foci of the locus are the middle points of the segments.

Discuss the case in which σ is an even multiple of $\frac{1}{2}\pi$.

[The difference of the sums of the angles which $PA_1, \ldots, PA_m$ and $PB_1, \ldots, PB_m$ make with a fixed line is σ. Now use Ex. 5, 6.

If $\sigma = k\pi$, we have the locus of Ex. 5.]

Ex. 11. If $A_1, \ldots, A_n$, $B_1, \ldots, B_{n-1}$ are fixed points, the locus of P such that $\quad PA_1 \,.\, PA_2 \,.\, \ldots \,.\, PA_n = k \,.\, PB_1^2 \,.\, PB_2 \,.\, \ldots \,.\, PB_{n-1}$

has $A_1, \ldots, A_n$ as ordinary foci.

The locus of P such that

$$PA_1 \,.\, PA_2 \,.\, \ldots \,.\, PA_n = k \,.\, PB_1 \,.\, PB_2 \,.\, \ldots \,.\, PB_{n-1}$$

has $A_1, \ldots, A_n$ as singular foci.

[In connexion with the above examples see Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques*, § 30.]

§ 3. Method of obtaining the Foci.

The singular foci are the intersections of the asymptotes which are circular lines. These asymptotes may be found in the usual manner. If one such asymptote is $x+iy = a+ib$, another is $x-iy = a-ib$, the curve being supposed real, and (a, b) is a real singular focus.

To obtain the ordinary foci, find the value of c such that $x+iy = c$ touches the curve, by putting $-1/c$ for λ and $-i/c$ for μ in the tangential equation, or otherwise. If $c = a+ib$ is such a value, a and b being real, (a, b) is a real focus.

Ex. 1. Find the real singular and ordinary foci of

(i) $2x(x^2+y^2) = a(3x^2-y^2)$.

(ii) $(x+y)(x^2+y^2)+2x(x-y) = 0$.

(iii) $x(x^2+y^2) = ay^2$.

(iv) $(x^2+y^2)^2-5x^2-4y^2+7 = 0$.

(v) $(x^2+y^2)^2-2c^2(x^2-y^2)+c^4-a^4 = 0$.

[(i) $(a, 0)$; $(-3a, 0)$.

(ii) $(0, 1)$; $(-3\mp2\sqrt{\sqrt{2}+1},\ -1\mp2\sqrt{\sqrt{2}-1})$.

(iii) $(-\frac{1}{2}a, 0)$; $(4a, 0)$.

(iv) $(\pm\frac{1}{2}, 0)$; $(\pm1, 0)$, $(\pm\sqrt{7}, 0)$.

(v) $(\pm c, 0)$; $(\pm(c^4-a^4)^{\frac{1}{2}}/c, 0)$ or $(0, \pm(a^4-c^4)^{\frac{1}{2}}/c)$ as $c>a$ or $c<a$.]

Ex. 2. Find the real foci of

(i) $(\lambda^2-2\mu^2)^2+\lambda\mu + 0$.

(ii) $4\lambda^2\mu^2-3\lambda^2+1 = 0$.

(iii) $2\lambda^2\mu+2\lambda\mu-2\mu^2+2\lambda+\mu+1 = 0$.

[(i) $(\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2})$, $(-\frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2})$.

(ii) $(\pm2, 0)$, $(0, \pm1)$.

(iii) $(1, \pm1)$, $(0, 1)$.]

Ex. 3. Given the tangential equation $\phi(\lambda, \mu) = 0$ of a curve in rectangular Cartesian coordinates, find an equation giving the distances of the real foci from the origin.

[If (r, θ) are the polar coordinates of a real focus,

$$\phi(-r^{-1}e^{-\theta i}, -ir^{-1}e^{-\theta i}) = 0 \quad\text{and}\quad \phi(-r^{-1}e^{\theta i}, ir^{-1}e^{\theta i}) = 0.$$

Now eliminate $e^{\theta i}$.]

Ex. 4. The product of the distances between a given real focus of that curve of a tangential pencil which touches the line at infinity and the real foci of any other curve of the pencil is constant.

Ex. 5. The distances from the pole of the intersections of a curve with its directrices are given by putting p infinite in its pedal equation.

Apply to the case of a conic.

Ex. 6. If $x = f(t)$, $y = \phi(t)$ is the parametral equation of a curve, the circular lines through the foci are the tangents at the points given by $f'(t) = \pm i\phi'(t)$.

Ex. 7. Find by means of Ex. 6 the foci of

(i) $x = at^2 + 2bt + c, \qquad y = At^2 + 2Bt + C.$

(ii) $x = at^p, \qquad y = at^{p+q}$, p and q being positive integers.

(iii) $x = at^2/(1+t^2), \qquad y = at^3/(1+t^2).$

Ex. 8. Find the singular foci, if any, of the curve in Ex. 6.

[Ch. III, § 9 shows the method of obtaining the tangents at the circular points. Apply it to Ex. 7 (iii).]

§ 4. Inverse and Reciprocal of Foci.

If S' is the point inverse to S with respect to a point O, $S'\omega'$ and $S'\omega$ are the inverses of $S\omega$ and $S\omega'$ respectively (Ch. I, § 10).

If S is a focus of a given curve, $S\omega$ and $S\omega'$ touch the curve. Hence $S'\omega'$ and $S'\omega$ touch the inverse curve. Therefore:

The inverses of the foci of a curve are the foci of the inverse curve.

The inverses of the lines joining ω' to the intersections of the curve with $O\omega$ are the tangents at ω to the inverse curve (see Ch. I, § 10, Ex. 3, or Ch. IX, § 1). Hence, if O is a focus of the curve, two tangents at ω to the inverse curve coincide. Hence:

The inverse of a curve with respect to a focus has cusps at ω *and* ω'.

As an example, the inverse with respect to O of a conic with real foci S and O is a limaçon

$$r = a + b\cos\theta,$$

which is a quartic having cusps at ω, ω' and a node at O. The real focus of the limaçon is the inverse of S.

If we reciprocate with respect to O (i.e. with respect to a circle whose centre is O), the lines $O\omega$ and $O\omega'$ become the points ω and ω'. Hence:

The reciprocal of a curve with respect to a focus is a curve through ω *and* ω', *and the reciprocal of the corresponding directrix is a singular focus.*

For instance, the reciprocal of a conic with respect to a focus is a circle, and the reciprocal of the corresponding directrix is the centre of the circle.

If we reciprocate with respect to any point O, the reciprocals of the foci are the lines joining the intersections of the reciprocal curve with $O\omega$ and $O\omega'$.

Ex. 1. The singular and ordinary foci H and S of the limaçon $r = a + b\cos\theta$ with node O are $(\frac{1}{2}b, 0)$ and $((b^2 - a^2)/2b, 0)$.

Ex. 2. $r(1-k^2) = a(\cos\theta - k)$, where k is a parameter, is a family of limaçons with a common node and focus.

Ex. 3. Trace the limaçon of Ex. 1 in the cases

$$b > a > 0, \quad a = b, \quad a > b > 0.$$

[O is a crunode, cusp, acnode; the curve being the inverse with respect to a focus of an hyperbola, parabola, ellipse.

There are real inflexions if a/b lies between 1 and 2; see Ch. IV, Fig. 1, and Ch. XVII, Fig. 1.

If $a = b$, the curve is called a *cardioid.*]

Ex. 4. If P is any point on the limaçon, OSP is twice the angle between OP and the tangent at P.

[Invert with respect to O. Then we get: 'The tangent to a conic makes equal angles with the focal distances of the point of contact.'

Obtain similarly other properties of the limaçon or cardioid.]

Ex. 5. In Ex. 1 $4HP^2 = 4bSP + 2a^2 - b^2$, P being any point of the curve.

[The equation with S as pole is

$$4b^2r^2 - 4br(a^2\cos\theta + b^2) + (a^2 - b^2)^2 = 0.]$$

Ex. 6. A limaçon is its own inverse with respect to the circle through the node with its centre at the ordinary focus.

[The equation of Ex. 5 is not altered on replacing r by $(a^2-b^2)^2/4b^2r$.]

Ex. 7. The angle ϕ between the tangent and radius vector of a limaçon is a maximum when $\phi = \theta$.

[$\tan\phi = -(a + b\cos\theta)/b\sin\theta$.]

Ex. 8. Find the locus of the inflexions and of the points of contact of the bitangent of the family $r = a + b\cos\theta$ when (i) b, (ii) a is kept constant and a or b respectively vary.

[(i) $r^2 + br\cos\theta + 2b^2\sin^2\theta = 0, \quad r + b\cos\theta = 0.$

(ii) $2r^2 + ar(3\cos^2\theta - 2) + 2a^2\sin^2\theta = 0, \quad 2r = a.$

For the inflexions $3ab\cos\theta + a^2 + 2b^2 = 0$, for the points of contact of the bitangent $2b\cos\theta + a = 0$.]

Ex. 9. The inverse of any curve with respect to a singular focus O has also a singular focus at O.

Ex. 10. If in § 4 S and O are reflexions of each other in the directrix corresponding to S, the inverse curve has inflexional tangents at ω and ω' which meet at S'.

CHAPTER VI

SUPERLINEAR BRANCHES

§ 1. Expansion of y in terms of x near the Origin.

LET $f(x, y) = 0$ be the equation of an algebraic curve passing through the origin.

In the neighbourhood of the origin it is possible to express the value of y on any branch of the curve through the origin (not touching $x = 0$ at the origin) in the form

$$y = ax + b\omega^{\beta} x^{\frac{\beta}{\alpha}} + c\omega^{\gamma} x^{\frac{\gamma}{\alpha}} + \quad . \quad . \quad . \quad . \quad . \qquad \text{(i)},$$

where $\omega^{\alpha} = 1$, $a, b, c, \dots$ are constants, and $\alpha, \beta, \gamma, \delta, \dots$ are positive integers in ascending order of magnitude.* The entire portion of the curve near the origin obtained by putting for ω every α-th root of unity in turn is called a *superlinear branch of order* α whose tangent is $y = ax$.

If we take a single α-th root of unity for ω we get a 'partial superlinear branch'.

If $\alpha = 1$, the branch is called *linear*. If we take the origin at any ordinary point of the curve, just one linear branch passes through the origin. Through an ordinary multiple point of order k with distinct tangents, k linear branches pass. It is only at multiple points where two or more tangents coincide that we can have superlinear branches.†

We assume the above results as proved in books on the 'Theory of Functions', and only give here the practical method of obtaining the expansions such as (i).

First suppose the curve has a linear branch through the origin. Take, for example, the curve

$$y - 2x + 2x^2 - 3xy - y^2 + 4x^3 = 0;$$

the method being general.

Substituting for y the expansion $ax + bx^2 + cx^3 + \dots$ in the equation of the curve, which we assume legitimate when x is sufficiently small, we have

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* For points on a curve near $(0, m)$ add m to the right-hand side of (i).

† We can also have multiple points with coincident tangents formed by the contact of distinct *linear* branches; see, for instance, Ch. III, Fig. 5.

$$(ax+bx^2+cx^3+\dots)-2x+2x^2-3x(ax+bx^2+cx^3+\dots)$$
$$-(ax+bx^2+cx^3+\dots)^2+4x^3=0.$$

Equating to zero the coefficients of $x, x^2, x^3, \dots$ in turn, we obtain $a=2$, $b=8$, $c=52, \dots$, and have as the required expansion

$$y=2x+8x^2+52x^3+\dots.\text{*}$$

As another example, consider the curve

$$y=Axy+By^2+Cx^3+Dx^2y+Exy^2+Fy^3+Gx^4+\dots,$$

which has an inflexion at the origin with $y=0$ as inflexional tangent.

Put $y=ax^2+bx^3+cx^4+\dots$ in this equation, and equate to zero the coefficients of $x^2, x^3, x^4, \dots$. We get $a=0$, $b=C$, $c=AC+G, \dots$.

Hence

$$y=Cx^3+(AC+G)x^4+\dots.$$

Similarly for a curve having r-point contact with $y=0$ at the origin we obtain an expansion of the form

$$y=ax^r+bx^{r+1}+cx^{r+2}+\dots.$$

Suppose now that the curve has any singularity at the origin and that $y=ax$ is a tangent at the origin. Putting $ax+\mathrm{y}$ for y in the equation of the curve, we obtain a curve touching the axis of x at the origin; x and y being considered as the current coordinates of any point on the curve. By the method given below we can expand y in terms of x and thence get y in terms of x. We may therefore confine ourselves to the case in which the curve touches the axis of x at the origin.

Suppose that Newton's diagram (Ch. III, § 3) gives us as an approximation to the curve touching $y=0$ at the origin terms represented by points which lie on a straight line making an angle $\tan^{-1} p/q$ with $x=0$, p and q being positive integers prime to one another.

In the equation $f(x, y)=0$ of the curve put

$$y=YX^p, \quad x=X^q.$$

We thus get a new curve $f(X^q, YX^p)=0$, if we consider X, Y as current coordinates. Suppose that $(0, m)$ is an intersection of the new curve with $X = 0$, and that through this point there is a linear branch of the curve. Then transferring the origin to $(0, m)$ we get an expansion $Y = m + bX + cX^2 + \dots$. Whence

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* Another method is to differentiate the equation of the curve repeatedly. Putting x and y zero, we obtain the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots$ at the origin. Then the expansion may be written down by Maclaurin's theorem.

$$y = x^{\frac{p}{q}}(m + bx^{\frac{1}{q}} + cx^{\frac{2}{q}} + \dots).$$

Replacing $x^{\frac{1}{q}}$ by $\omega x^{\frac{1}{q}}, \omega^2 x^{\frac{1}{q}}, \omega^3 x^{\frac{1}{q}}, \dots$, where ω is a primitive q-th root of unity, we get a complete superlinear branch of the original curve.

But if no linear branch of the new curve goes through its intersection $(0, m)$ with $X = 0$, we put in the original curve

$$y = Y + mX^p, \quad x = X^q,$$

so that our new curve is $f(X^q, Y + mX^p) = 0$.

We now apply Newton's diagram to this curve as before, in order to expand Y in terms of X and thence y in terms of x. Substitutions similar to the above must be repeated as often as necessary.

For example, suppose the curve has an ordinary cusp at the origin, the tangent at the cusp being $y = 0$. The curve is

$$y^2 = ax^3 + bx^2y + cxy^2 + dy^3 + ex^4 + \dots.$$

By an 'ordinary' cusp we mean that there is no relation between the coefficients $a, b, c, d, e, \dots$: in particular, none of them are zero.

Newton's diagram gives the approximation $y^2 = ax^3$ at the origin, and therefore put

$$x = X^2, \quad y = YX^3.$$

The curve becomes

$$Y^2 = a + eX^2 + bXY + cX^2Y^2 + \dots.$$

This has linear branches through $(0, \pm a^{\frac{1}{2}})$.

Putting

$$Y = \pm a^{\frac{1}{2}} + BX + CX^2 + \dots$$

into the equation of the curve and equating to zero the coefficients of $X, X^2, X^3, \dots$, we have

$$Y = \pm a^{\frac{1}{2}} + \tfrac{1}{2}bX \pm \tfrac{1}{2}a^{-\frac{1}{2}}(e + ca + \tfrac{1}{4}b^2)X^2 + \dots,$$

and thence

$$y = \pm a^{\frac{1}{2}}x^{\frac{3}{2}} + \tfrac{1}{2}bx^2 \pm \tfrac{1}{2}a^{-\frac{1}{2}}(e + ca + \tfrac{1}{4}b^2)x^{\frac{5}{2}} + \dots.$$

Exactly similarly for the curve

$$y^k = ax^{k+1} + bx^ky + cx^{k-1}y^2 + \dots,$$

which has an 'ordinary' superlinear branch of order k at the origin touching $y = 0$,* we get an expansion

$$y = A\omega x^{\frac{k+1}{k}} + B\omega^2 x^{\frac{k+2}{k}} + C\omega^3 x^{\frac{k+3}{k}} + \dots,$$

ω being any k-th root of unity.

If we have found the expansion

$$y = bx^{\frac{\beta}{\alpha}} + cx^{\frac{\gamma}{\alpha}} + \dots, \text{ where } \alpha < \beta < \gamma < \dots,$$

for a curve touching $y = 0$ at the origin, we may expand x in terms of y near the origin by putting

$$x = b^{-\frac{\alpha}{\beta}} y^{\frac{\alpha}{\beta}} (1 + A y^{\frac{1}{\beta}} + B y^{\frac{2}{\beta}} + \dots),$$

in the given expression for y in terms of x, expanding by the binomial theorem, and equating coefficients of powers of y on both sides of the resulting identity. Each possible value of $b^{-\frac{\alpha}{\beta}}$ gives an expression for x.

This process is known as 'reversion of series'.

In expanding y in terms of x we assumed that the curve did not touch $x = 0$ at the origin. If it does, we may use the preceding method to expand x in terms of y, and then obtain the expansion of y in terms of x by reversion of series.

The arithmetic of the methods described in this and the following section is often tedious. Devices for saving some of the labour will be described under the head of 'quadratic transformation' (Ch. IX, §§ 3 to 12).

Ex. 1. Find the expansion of the branch of the curve

$$(y - x^2)^2 = x^2 y^2 - y^4$$

near the origin

[Putting $y = YX^2$, $x = X$, we get $(Y-1)^2 = Y^2X^2 - Y^4X^4$. Putting $Y = 1 + aX + bX^2 + cX^3 + \dots$ and equating to zero coefficients of $X^2, X^3, \dots$, we have $a = \pm 1$, $b = 1$, $c = \pm\frac{1}{2}, \dots$. Hence the required expansion is $y = x^2(1 \pm x + x^2 \pm \frac{1}{2}x^3 + \dots)$.]

Ex. 2. Expand the branches of the following curves near the origin:

(i) $(y - x^2)^3 = x^2 y^3$.

(ii) $(y - x^3)^3 = x^2 y^3$.

(iii) $y - x^2 = x^2 y - 2y^3$.

(iv) $y^2 + 3x^2 y - xy^2 + 2x^4 = 0$.

(v) $y^3 - 3x^2 y^2 + x^4 y - x^5 = 0$.

(vi) $(y - x^3)^2 = x^5 y$.

(vii) $(y + x^2)^2 = 4y^2(x^2 + y^2)$.

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* 'Ordinary', because there is no special relation between the coefficients $a, b, c, \dots$; none of them, for instance, being zero.

[(i) $y = x^2 + \omega x^{\frac{8}{3}} + \omega^2 x^{\frac{10}{3}} + \omega^3 x^4 + \ldots$; where $\omega^3 = 1$.

(ii) $y = x^3 + \omega x^{\frac{11}{3}} + \omega^2 x^{\frac{13}{3}} + \omega^3 x^5 + \ldots$.

(iii) $y = x^2 + x^4 + \ldots$.

(iv) $y = x^2(-2 - 4x + 0x^2 + \ldots)$ and $y^2 = x^2(-1 + x - 3x^2 + \ldots)$.

(v) $y = x^{\frac{5}{3}}(1 + x^{\frac{1}{3}} + \frac{4}{3}x^{\frac{2}{3}} + \ldots)$ and two similar expansions.

(vi) $y = x^2(1 \pm x + \frac{1}{2}x^2 \pm \frac{1}{8}x^3 + \ldots)$.

(vii) $y = x^2(-1 \pm 2x - 4x^2 \pm 9x^3 + \ldots)$.]

Ex. 3. Find the expansion of the branch of the curve

$$(y - x^2)^2 = xy^3 - y^4$$

near the origin.

[The method of Ex. 1 gives a non-linear branch at (0, 1) after putting $y = YX^2$, $x = X$. Put then instead $y = Y + X^2$, $x = X$ and the curve becomes $Y^2 = X(Y + X^2)^3 - (Y + X^2)^4$ approximating to $Y^2 = X^7$ at the origin. Writing therefore $Y = \eta\xi^7$, $X = \xi^2$ we get

$$\eta^2 = (1 + \xi^3\eta)^3 - \xi^2(1 + \xi^3\eta)^4.$$

Put in this $\eta = 1 + a\xi + b\xi^2 + c\xi^3 + d\xi^4 + e\xi^5 + \ldots$ and compare coefficients. We get $a = 0$, $b = -\frac{1}{2}$, $c = \frac{3}{2}$, $d = -\frac{1}{8}$, $e = -2$. Hence

$$y = x^2 \pm x^{\frac{7}{2}} \mp \tfrac{1}{2}x^{\frac{9}{2}} + \tfrac{3}{2}x^5 \mp \tfrac{1}{8}x^{\frac{11}{2}} - 2x^6 + \ldots.]$$

Ex. 4. Find the expansion of the branch of $(y - x^3)^2 = y^3x^{10} + yx^9$ near the origin.

Ex. 5. Find x in terms of y if

(i) $y = 4x^2 + 8x^3 - 2x^4 + \ldots$.

(ii) $y = x^{\frac{3}{2}} + 3x^2 - x^{\frac{5}{2}} + \ldots$.

(iii) $8y = x^3 - 3x^4 - 3x^5 + \ldots$.

(iv) $y = x^{\frac{1}{2}} + 2x - 3x^{\frac{3}{2}} + \ldots$.

[(i) Put $x = \pm\frac{1}{2}y^{\frac{1}{2}}(1 + Ay^{\frac{1}{2}} + By + \ldots)$, expand, and compare coefficients. We get

$$x = \pm\tfrac{1}{2}y^{\frac{1}{2}} - \tfrac{1}{4}y \pm \tfrac{11}{32}y^{\frac{3}{2}} + \ldots.$$

(ii) Put $x = y^{\frac{2}{3}}(1 + Ay^{\frac{1}{3}} + By^{\frac{2}{3}} + \ldots)$. We get

$$x = y^{\frac{2}{3}} - 2y + \tfrac{23}{3}y^{\frac{4}{3}} + \ldots,$$

and the expansions obtained on replacing $y^{\frac{1}{3}}$ by $\omega y^{\frac{1}{3}}$ and $\omega^2 y^{\frac{1}{3}}$.

(iii) $x = 2y^{\frac{1}{3}}(1 + 2y^{\frac{1}{3}} + 16y^{\frac{2}{3}} + \ldots)$.

(iv) $x = y^2(1 - 4y + 26y^2 + \ldots)$.]

Ex. 6. A curve has a rhamphoid cusp at the origin with $y = 0$ as tangent. Show that near the origin

$$y = ax^2 \pm bx^{\frac{5}{2}} + cx^3 \pm dx^{\frac{7}{2}} + \ldots.$$

[Cf. Ch. III, § 8, Ex. 2.]

§ 2. Intersections of Two Curves.

We showed in Ch. I, § 7, that two curves of degrees n, N

$$\left.\begin{aligned} f(x, y) &\equiv a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \ldots + a_n \\ &\equiv (y-u_1)(y-u_2)\ldots(y-u_n) = 0 \\ F(x, y) &\equiv b_0 y^N + b_1 y^{N-1} + b_2 y^{N-2} + \ldots + b_N \\ &\equiv (y-v_1)(y-v_2)\ldots(y-v_N) = 0 \end{aligned}\right\}$$

meet in nN points.

In this a_r, b_r denote polynomials of the r-th degree in x; while $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_N$ are functions of x, of which we have obtained the expansion (§ 1), when x is sufficiently small.

We assume that $x = 0$ meets the curves in finite points only and does not touch either curve, also that no intersection of the curves other than the origin lies on $x = 0$.

The result of eliminating y between the equations of the two curves may be expressed by equating to zero a certain determinant (Ch. I, § 7), or in the form

$$\begin{aligned} \phi(x) \equiv (u_1 - v_1)(u_1 - v_2)\ldots(u_1 - v_N) \times (u_2 - v_1)(u_2 - v_2)\ldots(u_2 - v_N) \\ \times \ldots \times (u_n - v_1)(u_n - v_2)\ldots(u_n - v_N) = 0. \end{aligned}$$

In this equation $\phi(x)$ is a polynomial of degree nN in x.* Suppose we wish to find the number of zero roots, i. e. the number of intersections of the given curves which coincide with the origin.

The number is ϵ, where x^ϵ is the lowest power of x in $\phi(x)$, i. e. the product of the lowest powers of x in each of the expressions

$$(u_i - v_j) \quad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, N.$$

It is evident that the lowest power of x in $u_i - v_j$ is x^0 unless $y = u_i$, and $y = v_j$ are (partial) branches through the origin. Hence:

The number of intersections of two given curves which coincide with the origin is the index of the product of the lowest powers of x in all possible expressions of the form $u_i - v_j$; $y = u_i$ and $y = v_j$ being expansions of y in terms of x for branches of the two curves passing through the origin.

Let us consider the following examples, which will be useful later on.†

(i) One curve has a node or cusp at the origin, and the other passes through the origin but does not touch it there.

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* It is the determinant just mentioned multiplied by a constant.

† Cases (i), (ii), (v) have been already dealt with in Ch. I, § 7.

In this case
$u_1 = ax + bx^2 + \dots,\ u_2 = a'x + b'x^2 + \dots;\ v_1 = Ax + Bx^2 + \dots.$

Hence $\Pi(u_i - v_j) \equiv \{(a-A)x + \dots\}\{(a'-A)x + \dots\}$, and the curves meet twice at the origin, as might have been anticipated.

(ii) One curve has a cusp at the origin, and the other passes through the cusp and touches the first curve there.

Here
$u_1 = ax^{\frac{3}{2}} + bx^2 + \dots,\ u_2 = -ax^{\frac{3}{2}} + bx^2 - \dots;\ v_1 = Ax^2 + Bx^3 + \dots:$
and the curves meet thrice at the origin.

(iii) The curves have a node at the origin, the tangents at the node to the two curves being the same.

Here

$$u_1 = ax + bx^2 + \dots.\quad u_2 = a'x + b'x^2 + \dots;$$
$$v_1 = ax + Bx^2 + \dots,\quad v_2 = a'x + B'x^2 + \dots.$$

Hence

$$\Pi(u_i - v_j) \equiv \{(a-a')x + \dots\}\{(a'-a)x + \dots\}$$
$$\{(b-B)x^2 + \dots\}\{(b'-B')x^2 + \dots\};$$

and the curves meet six times at the origin.

(iv) One curve has a cusp at the origin, while the other has a triple point, two tangents at which coincide with the cuspidal tangent.

Here

$$u_1 = ax^{\frac{3}{2}} + \dots,\quad u_2 = -ax^{\frac{3}{2}} + \dots;\quad v_1 = bx + cx^2 + \dots,$$
$$v_2 = Ax^{\frac{3}{2}} + \dots,\ v_3 = -Ax^{\frac{3}{2}} + \dots,\ \text{or}\ v_2 = Bx^2 + \dots,\ v_3 = Cx^2 + \dots.$$

In either case the curves meet eight times at the origin.

(v) The curves have k and K linear branches through the origin, no two of the $k+K$ tangents coinciding.

Here $\Pi(u_i - v_j)$ has kK factors each of the type

$$ax + bx^2 + cx^3 + \dots;$$

and the curves meet kK times at the origin.*

(vi) The curves have k linear branches through the origin with the same tangents for each curve.

Here $k(k-1)$ of the factors in $\Pi(u_i - v_j)$ are of the type

$$ax + bx^2 + cx^3 + \dots$$

and k are of the type

$$Bx^2 + Cx^3 + \dots.$$

The curves meet $k(k+1)$ times at the origin.

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* This result might have been expected; for each of the k branches of one curve at the origin meets each of the K branches of the other curve once at the origin.

Ex. 1. The tangent at the origin to the superlinear branch (i) of § 1 meets it in β points coinciding with the origin.

Ex. 2. Two curves have linear branches through the origin. Show that, if the expansions of y in terms of x for the two curves are identical as far as the terms in x^r, the curves have $(r+1)$-point contact at the origin.

Ex. 3. The expansion of y in terms of x for the conic

$$y = mx + ax^2 + 2hx(y - mx) + b(y - mx)^2$$

is $$y = mx + ax^2 + 2ahx^3 + a(4h^2 + ab)x^4 + 2ah(4h^2 + 3ab)x^5 + \ldots.$$

Ex. 4. Use Ex. 2 and 3 to obtain the conic of closest contact at the origin with $x - y + xy + x^3 = 0$.

[For the curve $y = x + x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots$. Comparing the coefficients of x, x^2, x^3, x^4 in this and the expansion of Ex. 3 we have

$$m = 1, \quad a = 1, \quad h = 1, \quad b = -2.$$

Other methods are (i) to differentiate the equations of curve and conic four times and to identify the values of

$$y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$$

for curve and conic at the origin; (ii) to find the lines joining the intersections of curve and conic to the origin and to choose m, a, h, b so that five of them are $y = x$.]

Ex. 5. Find the conics of closest contact at the origin with

$$a^2x + by^2 + y^3 = 0 \quad \text{and} \quad x^3 + y^3 = 3axy.$$

$$[b^3(a^2x + by^2) = (a^4x + a^2b^2y)x \quad \text{and} \quad y^2 = 3ax, \quad x^2 = 3ay.]$$

Ex. 6. Find the parabolas of closest contact at the origin with

$$0 = x + y + xy + x^3 \quad \text{and} \quad y = x + x^2 - 2x^3.$$

$$[x + y = y^2 \quad \text{and} \quad y - x = (2x - y)^2.]$$

Ex. 7. Two curves have linear branches touching the same tangent at O, one having p-point and the other q-point contact with the tangent, where $p \geqslant q$. Show that they meet q times at O.

[See Ex. 2 and § 1. See also Ch. IX, § 12, Ex. 11.]

Ex. 8. A curve has a k-ple point O with superlinear branches of order $r_1, r_2, \ldots, r_l$ having distinct tangents ($k = \Sigma r$). Another curve has at O a $(k-1)$-ple point such that the tangent to the branch of order r_t of the first curve is a tangent to a branch of order $r_t - 1$ of the second. How many of their intersections coincide at O?

$$[\Sigma r\{(r-1)(r+1)/r + (k-1) - (r-1)\} = k^2 - l.]$$

§ 3. Tangential Equation near a Point.

Suppose that it is required to obtain an approximation to the tangential equation of a branch of a curve near a point. We may of course find the tangential equation of the curve as in Ch. IV, § 3, and then employ a method similar to that used in § 1 for point-coordinates. But a less laborious process will usually be the following.

Suppose the branch touches $y = 0$ at the point $(h, 0)$. We want the condition that $\lambda x + \mu y + \nu = 0$ should touch the branch. We shall take $\mu = 1$, and obtain the condition by expressing ν in terms of λ, just as in § 1 we expressed y in terms of x.

Transfer the origin to the point $(h, 0)$. The line becomes $\lambda x + \mu y + \nu + h\lambda = 0$, and we have the point-equation of the branch by § 1 in the form

$$y = b\omega^{\beta} x^{\frac{\beta}{\alpha}} + c\omega^{\gamma} x^{\frac{\gamma}{\alpha}} + d\omega^{\delta} x^{\frac{\delta}{\alpha}} + \dots \quad . \quad . \quad . \quad . \quad \text{(i)}$$

The tangent at the point (ξ, η) of the curve is

$$y - \eta = \frac{d\eta}{d\xi}(x - \xi)$$

or, substituting

$$b\omega^{\beta} \xi^{\frac{\beta}{\alpha}} + c\omega^{\gamma} \xi^{\frac{\gamma}{\alpha}} + d\omega^{\delta} \xi^{\frac{\delta}{\alpha}} + \dots$$

for η and writing $\xi = X^{\alpha}$, it is

$$(b\beta\omega^{\beta} X^{\beta-\alpha} + c\gamma\omega^{\gamma} X^{\gamma-\alpha} + d\delta\omega^{\delta} X^{\delta-\alpha} + \dots)\, x - \alpha y$$
$$= b(\beta - \alpha)\,\omega^{\beta} X^{\beta} + c(\gamma - \alpha)\,\omega^{\gamma} X^{\gamma} + \dots .$$

Comparing with $\lambda x + \mu y + \nu + h\lambda = 0$, we have on putting $\mu = 1$ and $\lambda = \Lambda^{\beta-\alpha}$

$$\left.\begin{aligned} \alpha\Lambda^{\beta-\alpha} &= -(b\beta\omega^{\beta} X^{\beta-\alpha} + c\gamma\omega^{\gamma} X^{\gamma-\alpha} + \dots) \\ \alpha(\nu + h\Lambda^{\beta-\alpha}) &= b(\beta - \alpha)\,\omega^{\beta} X^{\beta} + c(\gamma - \alpha)\,\omega^{\gamma} X^{\gamma} + \dots \end{aligned}\right\} \quad . \quad . \text{(ii)}.$$

Substituting in the first of these equations

$$X = A\Lambda + B\Lambda^{2} + C\Lambda^{3} + \dots \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

and equating to zero the coefficients of powers of Λ, we get $A, B, C, \dots$ in turn.

Then substitute the value of X given by (iii) in the second of equations (ii) and we have ν expressed in terms of Λ, which is $\lambda^{\frac{1}{\beta-\alpha}}$.

If we now replace λ by x and ν by y, we get the point-equation of the polar reciprocal of the branch (i) with respect to the base-conic $x^2 + 2y = 0$ (Ch. IV, § 6, Ex. 1).

For example, in the neighbourhood of a cusp at the origin at which $y = 0$ is the tangent, we have

$$y = \pm a x^{\frac{3}{2}} + b x^{2} \pm c x^{\frac{5}{2}} + d x^{3} \pm \dots .$$

The above process gives now

$$\left.\begin{aligned} \lambda &= \mp \tfrac{3}{2} aX - 2bX^{2} \mp \tfrac{5}{2} cX^{3} - \dots \\ \nu &= \pm \tfrac{1}{2} aX^{3} + bX^{4} \pm \tfrac{3}{2} cX^{5} + \dots \end{aligned}\right\},$$

where $\xi = X^{2}$.

Putting $X = A\lambda + B\lambda^2 + C\lambda^3 + \dots$ in the first of these equations, and equating to zero the coefficients of $\lambda, \lambda^2, \lambda^3, \dots$ we get

$$X = \mp \frac{2}{3a}\lambda \mp \frac{16b}{27a^3}\lambda^2 \pm \dots \cdot$$

whence

$$\nu = -\frac{4}{27a^2}\lambda^3 - \frac{16b}{81a^4}\lambda^4 + \dots.$$

The polar reciprocal of this branch with respect to $x^2 + 2y = 0$ is

$$y = -\frac{4}{27a^2}x^3 - \frac{16b}{81a^4}x^4 + \dots.$$

This equation represents a linear branch with an inflexion at the origin, verifying the result of Ch. IV, § 7, that to a cusp of a curve corresponds an inflexional tangent of the polar reciprocal.

Ex. 1. Expand ν in terms of λ for the curves of § 1, Ex. 2, (i) to (iii).

Ex. 2. Show that, if

$$y = a_2x^2 + a_3x^3 + a_4x^4 + \dots \quad \text{and} \quad \nu = b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + \dots$$

represent the same linear branch touching $y = 0$ at the origin, b_r involves $a_2, a_3, \dots, a_r$, but not $a_{r+1}, a_{r+2}, \dots$.

§ 4. Common Tangents of Two Curves.

As in § 2, we prove that:

The number of those common tangents to two curves, which coincide with the axis of x, is the index of the product of the lowest powers of λ in expressions of the form $u_i - v_j$, $\nu = u_i$ and $\nu = v_j$ being expansions of ν in terms of λ for branches of the two curves touching the axis of x.

We may also find the number of common tangents by reciprocation.

Ex. 1. Two curves have superlinear branches of orders p and q $(p \not> q)$ with a common tangent l. How many of their common tangents coincide with l?

[One or q according as the point-singularities do not or do coincide. Reciprocate and use § 2, Ex. 7.]

Ex. 2. Show that, if two linear branches have r-point contact at P, r of their common tangents coincide with the tangent at P.

[See § 2, Ex. 2, and § 3, Ex. 2.]

§ 5. *Polar Reciprocal of Superlinear Branch.

If the polar reciprocal of a superlinear branch of order α at O whose tangent meets it in β points coinciding with O is a superlinear branch of order α' at O' whose tangent meets it in β' points coinciding with O', then

$$\alpha + \alpha' = \beta = \beta'.$$

Take the superlinear branch (i) of § 3. It is seen at once by § 2 that its tangent meets it in β points coinciding with the origin.

The polar reciprocal is obtained by putting x, y for λ, ν in the tangential equation of the branch (§ 3).

From equations (ii) and (iii) of § 3 we have $\nu + h\lambda$ expressed in ascending powers of Λ, the lowest power being Λ^{β}; where $\lambda = \Lambda^{\beta-\alpha}$. Hence $\alpha' = \beta - \alpha$ and $\beta' = \beta$, unless the index of every power of Λ in the expansion of $\nu + h\lambda$ is divisible by a factor of $\beta - \alpha$, in which cases $(\beta-\alpha)/\beta = \alpha'/\beta'$ and $\beta - \alpha > \alpha'$, $\beta > \beta'$.

But considering the original curve to be obtained by a second reciprocation from the reciprocal, we should obtain

$$(\beta' - \alpha')/\beta' = \alpha/\beta \text{ and } \beta' - \alpha' \geqslant \alpha,\ \beta' \geqslant \beta.$$

This is inconsistent with the preceding, and therefore

$$\alpha + \alpha' = \beta = \beta'.$$

We deduce at once that:

The polar reciprocal of a superlinear branch of order α is in general a linear branch having $(\alpha+1)$-point contact with its tangent.

For taking the tangent to the superlinear branch at the origin as axis of x, the Cartesian equation of the curve takes the form

$$y^{\alpha} u_{k-\alpha} = u_{k+1} + u_{k+2} + \dots;$$

where u_r is homogeneous in x and y of degree r. Putting $y = 0$ we get an equation of degree $k+1$ for x in general. But the origin is a multiple point of order k, so that the tangent to the superlinear branch meets the other branches in $k - \alpha$ points at the origin. Therefore β of the theorem at the beginning of this section is $\alpha + 1$. Hence $\alpha' = 1$, $\beta' = \alpha + 1$; as was to be proved.

The theorem may also be established directly by the process of § 3.

For example, the polar reciprocal of an inflexion is a cuspidal tangent, of a point of undulation is the tangent to a superlinear branch of the third order, &c. (cf. Ch. IV, § 7).

The number of tangents from O to a superlinear branch at O which coincide with the tangent at O is equal to the number of intersections of the reciprocal branch with its tangent which coincide with the singularity O'.

Also the number of tangents from a point on the tangent at O which coincide with this tangent is equal to the number of intersections of a line through O' with the reciprocal branch which coincide with O'.

Hence the relation $\alpha+\alpha' = \beta = \beta'$ gives us

If a superlinear branch of order α at O meets its tangent in β points coinciding with O, β tangents from O to the branch coincide with the tangent at O, and $\beta-\alpha$ tangents to the branch from any point on the tangent at O coincide with the tangent at O.

Ex. 1. Suppose a curve has a superlinear branch of order 2 at the origin with $y = 0$ as tangent and expansion

$$y = ax^2 \pm bx^{\frac{5}{2}} + cx^3 \pm dx^{\frac{7}{2}} + \ldots.$$

Then $\alpha = 2$, $\beta = 4$, so that $\alpha' = 2$, $\beta' = 4$. The origin is a *rhamphoid cusp* (§ 1, Ex. 6). We see that the reciprocal singularity is of the same type as the original.

Ex. 2. If α is the 'order' and $\beta-\alpha$ the 'class' of a superlinear branch, the order and class of a branch are respectively the class and order of the reciprocal branch.

[For the nomenclature see Halphen, *Bull. de la Soc. Math. de France*, vi (1877), p. 10.]

Ex. 3. Write down the order and class of the branches in § 1, Ex. 2 to 5.

CHAPTER VII

POLAR CURVES

§ 1. Polar Curves.

LET O be any fixed point. Through O draw any line OP meeting a given curve of the n-th degree in $Q_1, Q_2, \ldots, Q_n$. The locus of P such that

$$\frac{OQ_1}{PQ_1}+\frac{OQ_2}{PQ_2}+\ldots+\frac{OQ_n}{PQ_n}=0$$

is called the *first polar curve* of O with respect to the given curve.*

Similarly the locus of P such that

$$\Sigma OQ_i . OQ_j/PQ_i . PQ_j = 0 \quad (i, j = 1, 2, \ldots, n\,;\ i \neq j)$$

is called the *second polar curve* of O with respect to the given curve; the locus of P such that

$$\Sigma OQ_i . OQ_j . OQ_k/PQ_i . PQ_j . PQ_k = 0$$
$$(i, j, k = 1, 2, \ldots, n\,;\ j \neq k,\ k \neq i,\ i \neq j)$$

is called the *third polar curve* of O with respect to the given curve; and so on.

We shall see in § 2 that the first, second, third, ..., $(n-2)$-th, $(n-1)$-th polar curves are of degrees $n-1$, $n-2$, $n-3$, ..., 2, 1 respectively. They are therefore also called the polar $(n-1)$-ic, $(n-2)$-ic, $(n-3)$-ic, ..., conic, line of O with respect to the given curve.

If l is the polar line of O, O is called a 'pole' of l with respect to the curve.

If we project the curve from a vertex V, the k-th polar curve of O (for each value of k) projects into the k-th polar curve of the projection of O with respect to the projection of the given curve.

---

* Or 'first polar of O for the given curve', if there is no ambiguity; and so for 'second polar', &c.

For in the figure, denoting projections of points by dashes,

$$\frac{OQ}{PQ}=\frac{VO\,.\sin OVQ\,.\operatorname{cosec} VQP}{VP\,.\sin PVQ\,.\operatorname{cosec} VQP}=\frac{VO\,.\sin OVQ}{VP\,.\sin PVQ},$$

and similarly

$$\frac{O'Q'}{P'Q'}=\frac{VO'.\sin OVQ}{VP'.\sin PVQ}.$$

Hence

$$\frac{OQ}{PQ}=\frac{O'Q'}{P'Q'}\times\frac{VO\,.\,VP'}{VO'\,.\,VP}.$$

Therefore, if

$$\Sigma\, OQ_1\,.\,OQ_2\,.\,\ldots\,.\,OQ_k/PQ_1\,.\,PQ_2\,.\,\ldots\,.\,PQ_k=0,$$

we have

$$\Sigma\, O'Q_1'\,.\,O'Q_2'\,.\,\ldots\,.\,O'Q_k'/P'Q_1'\,.\,P'Q_2'\,.\,\ldots\,.\,P'Q_k'=0.$$

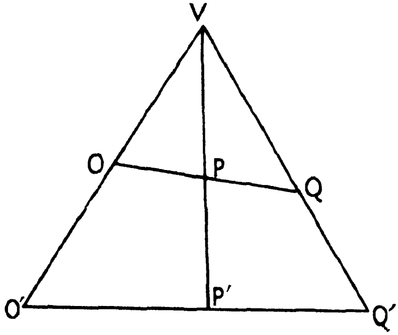

Fig. 1.

The locus of points whose polar conics degenerate into a pair of straight lines is called the *Hessian* of the given curve.

The theorem just proved shows that

The projection of the Hessian of a curve is the Hessian of the projection of the curve.

§ 2. Equation of Polar Curves.

In § 1 let O be (X, Y, Z) and P be (x, y, z), while the equation of the curve is $f(x, y, z)=0$. The point Q dividing OP so that $OQ/QP=\lambda/\mu$ is

$$\left(\frac{\lambda x+\mu X}{\lambda+\mu},\ \frac{\lambda y+\mu Y}{\lambda+\mu},\ \frac{\lambda z+\mu Z}{\lambda+\mu}\right).$$

If this lies on the curve,

$$f(\lambda x+\mu X,\ \lambda y+\mu Y,\ \lambda z+\mu Z)=0,$$

i. e.

$$\lambda^n f(x,y,z)+\frac{1}{1\,!}\lambda^{n-1}\mu\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)f$$
$$+\frac{1}{2\,!}\lambda^{n-2}\mu^2\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^2 f+\ldots$$
$$+\frac{1}{n\,!}\mu^n\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^n f=0,$$

or

$$\mu^n f(X,Y,Z)+\frac{1}{1\,!}\mu^{n-1}\lambda\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)f$$
$$+\frac{1}{2\,!}\mu^{n-2}\lambda^2\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^2 f+\ldots$$
$$+\frac{1}{n\,!}\lambda^n\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^n f=0,$$

by Taylor's theorem.

In this $\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^k f$ means

$$\Sigma a_{pqr}X^pY^qZ^r\frac{\partial^k f(x,y,z)}{\partial x^p\,\partial y^q\,\partial z^r},\quad (k=p+q+r),$$

where $$(x+y+z)^k\equiv\Sigma a_{pqr}x^py^qz^r.$$

Also $\left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^k f$ means the same thing with x, y, z, X, Y, Z put for X, Y, Z, x, y, z respectively.

If Q lies on the k-th polar curve, the sum of the products of the roots taken k at a time of the equation in λ/μ is zero. Hence the equation of the k-th polar curve is

$$\left(X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}+Z\frac{\partial}{\partial z}\right)^k f=0,\ \text{ or }\ \left(x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y}+z\frac{\partial}{\partial Z}\right)^{n-k} f=0,$$

the two equations being of course identical.

We have at once

The k-th polar curve of an n-ic is an $(n-k)$-ic.

To find the polar curves, if the equation of the curve is given in Cartesian coordinates as $f(x,y)=0$, we find the polar curves of $f(x/z,\ y/z)=0$ by the above method, and then put $z=1$. (See Ch. I, § 3.)

Ex. 1. The k-th polar curves of the vertices of the triangle of reference are

$$\frac{\partial^k f}{\partial x^k}=0, \qquad \frac{\partial^k f}{\partial y^k}=0, \qquad \frac{\partial^k f}{\partial z^k}=0.$$

Ex. 2. Any polar curve of the 'triangular-symmetric' curve

$$(x/a)^n+(y/b)^n+(z/c)^n=0$$

is triangular-symmetric.

Ex. 3. If a curve has a 'centre' O,* the polar curves of O and the polar curves of any point at infinity have O as a centre.

Ex. 4. The r-th polar curve of O is the s-th polar curve of O with respect to the $(r-s)$-th polar curve.

Ex. 5. If P lies on the k-th polar curve of Q for an n-ic, Q lies on the $(n-k)$-th polar curve of P.

[The polar curves are

$$\left(x_2\frac{\partial}{\partial x}+\dots\right)^k f=0 \qquad \text{and} \qquad \left(x\frac{\partial}{\partial x_1}+\dots\right)^k f=0,$$

P and Q being (x_1, y_1, z_1) and (x_2, y_2, z_2).

An alternative method in Ex. 5, 6, 7 is to take P and Q as vertices of the triangle of reference]

Ex. 6. If Q is a node of the $(n-k)$-th polar of P for an n-ic, P is a node of the $(k-1)$-th polar of Q.

Ex. 7. The k-th polar of P with respect to the r-th polar of Q for $f=0$ is also the r-th polar of Q with respect to the k-th polar of P.

[It is $\left(x_1\frac{\partial}{\partial x}+\dots\right)^k\left(x_2\frac{\partial}{\partial x}+\dots\right)^r f=0.$]

Ex. 8. If the polar conic of O with respect to a given cubic has ABC as a self-conjugate triangle, the polars of A with respect to the polar conics of B and C, &c., must be concurrent at O.

If B and C are given, there is in general one position of A and O.

[Take ABC as triangle of reference.]

Ex. 9. If P lies on a given line, the envelope of the polar line of P with respect to a given n-ic is of class $n-1$.

[Take the line as $z=0$ and find the number of tangents from $(0, 0, 1)$. More generally, if P lies on a given N-ic, the class is $N(n-1)$.]

Ex. 10. If the asymptotes of a curve are parallel to the sides of a regular polygon, so are the asymptotes of the polar curves of any infinitely distant point.

Ex. 11. On the polar line of O there are three points whose first polar curves have an inflexion at O.

[Take the curve as $z^n+z^{n-2}u_2+z^{n-3}u_3+\dots+u_n=0$, for which $z=0$ is the polar line of $(0, 0, 1)$. See Ex. 5.]

Ex. 12. (i) If the polar conic of O with respect to a given n-ic has a self-conjugate triangle which is inscribed in a given conic, the locus of O is an $(n-2)$-ic.

(ii) If a triangle can be inscribed in the polar conic of O which is self-conjugate with respect to a given conic, the locus of O is a $2(n-2)$-ic (cf. § 9, Ex. 9).

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* The curve is brought into self-coincidence by rotation through 180° about O.

[See Salmon's *Conic Sections*, §§ 373, 375. The reader may consider the cases in which the conics touch, or a triangle can be inscribed to one and circumscribed to the other, &c.]

Ex. 13. Through a point O any line is drawn meeting a given n-ic in $Q_1, Q_2, \ldots, Q_n$. Show that an 'axial' direction of the line making $OQ_1 \cdot OQ_2 \ldots\ldots OQ_n$ a maximum or a minimum is perpendicular to the polar line of the point at infinity in this direction.

Show that there are n such axial directions in general, and that they are independent of the position of O, and are the same for all n-ics with the same infinite points.

Discuss the case in which the curve passes through the circular points any number of times.

[Use polar coordinates. Note the case $n = 2$, and the case of an n-ic only meeting the line at infinity at the circular points. See *Bull. de la Soc. Math. de France*, ix, p. 49.]

Ex. 14. The k-th polar curve of P with respect to an n-ic having an $(n-1)$-ple point at O is an $(n-k)$-ic having an $(n-k-1)$-ple point at O.

If two of the tangents at O to the polar curve coincide (or are perpendicular), the locus of P is straight lines through O.

§ 3. Polar Curves of a Point on the Curve.

Suppose the triangle of reference chosen so that O is $(0, 0, 1)$. Then the k-th polar curve for $f = 0$ is $\dfrac{\partial^k f}{\partial z^k} = 0$ by § 2.

If the given curve is

$$0 = az^n + (b_0 x + b_1 y)\, z^{n-1} + (c_0 x^2 + 2c_1 xy + c_2 y^2)\, z^{n-2} + \ldots \quad . \quad \text{(i)},$$

the k-th polar curve of O is

$$0 = \frac{n!\, a}{(n-k)!} z^{n-k} + \frac{(n-1)!}{(n-k-1)!} (b_0 x + b_1 y)\, z^{n-k-1}$$
$$+ \frac{(n-2)!}{(n-k-2)!} (c_0 x^2 + 2c_1 xy + c_2 y^2)\, z^{n-k-2} + \ldots .$$

If O lies on the given curve, $a = 0$; and the tangent at O is $b_0 x + b_1 y = 0$. Hence the polar curves of O all touch the given curve at O. And, more generally, we show in the same manner that

Any polar curve of an r-ple point O of a given curve has an r-ple point at O with the same tangents as the given curve.

If O is an inflexion of the given curve, $a = 0$ and $b_0 x + b_1 y$ is a factor of $c_0 x^2 + 2c_1 xy + c_2 y^2$.

Hence O is an inflexion of all the polar curves; and similarly in general:

The polar curves of a point O at which the tangent has r-point contact have r-point contact at O with the same tangent.

Ex. 1. The $(n-r)$-th polar curve of an r-ple point of an n-ic consists of the r tangents at the point.

The $(n-r+1)$-th, $(n-r+2)$-th, ... polar curves are non-existent.

Ex. 2. The ratio of the curvature at O of an n-ic to the curvature of the k-th polar curve of O at O is $(n-1)/(n-k-1)$.

[If the curve is $0 = y + ax^2 + \dots$, the k-th polar of the origin is

$$0 = (n-1)\,y + (n-k-1)\,ax^2 + \dots .]$$

Ex. 3. The polar conic of (ξ, η, ζ) with respect to

$$(b-c)\,(\eta z - \zeta y)\,yz + (c-a)\,(\zeta x - \xi z)\,zx + (a-b)\,(\xi y - \eta x)\,xy = 0$$

has the triangle of reference as a self-conjugate triangle.

[It is $(a\xi^2 + b\eta^2 + c\zeta^2)\,(x^2 + y^2 + z^2) = (\xi^2 + \eta^2 + \zeta^2)\,(ax^2 + by^2 + cz^2)$.]

Ex. 4. The locus of a point whose polar conic with respect to a given n-ic is a rectangular hyperbola is a $(n-2)$-ic.

[If the equation of the curve in rectangular Cartesian coordinates is $f(x, y) = 0$, the locus is $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.]

Ex. 5. The number of points on an n-ic whose polar conic is a rectangular hyperbola is $n\,(n-2)$.

Ex. 6. The locus of a point whose polar conic with respect to a given n-ic is a parabola is a $2\,(n-2)$-ic.

[The locus is $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$.]

Ex. 7. The number of points on an n-ic whose polar conic with respect to the curve is a parabola is in general $2n\,(n-2) - 2\kappa$, where κ is the number of cusps.

[The locus of Ex. 6 passes through each cusp of the n-ic, as is seen by taking the origin at the cusp and writing down the equation of the locus. The polar conic of a cusp is the cuspidal tangent twice over by Ex. 1, which we do not count as a parabola.]

Ex. 8. The locus of a point whose polar conic with respect to a given n-ic has given eccentricity ($\neq 0$, 1, or $2^{\frac{1}{2}}$) is a $2\,(n-2)$-ic; and the number of points on an n-ic whose polar conic with respect to the curve has a given eccentricity is $2n\,(n-2)$.

Ex. 9. There are $(n-2)^2$ points in the plane of an n-ic whose polar conic is a circle.

[$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ are equations giving the points.]

Ex. 10. If all the polar conics for

$$f(x, y) \equiv u_n + u_{n-1} + \dots + u_1 + u_0 = 0$$

are rectangular hyperbolas, where u_k is homogeneous of degree k in x and y, then $u_k = 0$ are lines parallel to the sides of a regular polygon.

The n asymptotes meet in a point and are parallel to the sides of a regular polygon; and so are the tangents at any multiple point.

$$\Big[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \equiv \sum r^{k-2}\left(\frac{\partial^2}{\partial \theta^2} + k^2\right)(u_k r^{-k}) \equiv 0.$$

Therefore $$u_k = r^k\,(a \cos k\theta + b \sin k\theta).$$

If one asymptote passes through the origin, u_n and u_{n-1} have a common factor. If two asymptotes meet at the origin, $u_{n-1} \equiv 0$.]

Ex. 11. All the polar conics for $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ are parabolas; and conversely every curve all of whose conics are parabolas is of this type.

The curve is of degree and class n. It has a superlinear branch of order $n-1$ at $(0, \infty)$, the line at infinity being the tangent.

The centroid of the intersections of this curve with any other lies on a fixed line, if $n > 2$.

[Choose axes of reference for the curve of Ex. 10 such that x is a factor of u_n. Then substitute in

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \equiv \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

The centroid lies on $x = 0$, if $a_{n-1} = 0$.]

Ex. 12. If every polar curve of degree r goes through two fixed points A and B, the curve is a $(2r-2-k)$-ic with a $(r-1-k)$-ple point at A and B.

[Taking A and B as $(1, 0, 0)$ and $(0, 1, 0)$,

$$\frac{\partial^r f}{\partial x^r} \equiv 0 \quad \text{and} \quad \frac{\partial^r f}{\partial y^r} \equiv 0;$$

$f = 0$ being the equation of the curve.

Taking $r = 2$ or 3, we have the two following examples.]

Ex. 13. If every polar conic is a circle, the curve is a circle.

Ex. 14. If every polar cubic is circular, the curve is a circular cubic or bicircular quartic.

Ex. 15. If a line meets a $(2n+1)$-ic in $P_1, P_2, \ldots, P_{2n+1}$, and we derive a series of curves $\sigma_1, \sigma_2, \ldots, \sigma_{2n}$, such that σ_1 is the first polar of P_1 for the given curve, σ_2 is the first polar of P_2 for σ_1, σ_3 is the first polar of P_3 for σ_2, ..., then the line σ_{2n} goes through P_{2n+1}.

[Take the line as $z = 0$.]

Ex. 16. The k-th polar curves of a point O with respect to a pencil of n-ics form a pencil of $(n-k)$-ics.

Ex. 17. The tangent at O to that curve of a given pencil which passes through O goes through the intersection of the polar lines of O with respect to any two curves of the pencil.

[Take $k = n-1$ in Ex. 16.]

Ex. 18. The locus of the poles of a given line with respect to all curves of a pencil of n-ics is a $2(n-1)$-ic passing through the points of contact of those curves of the pencil which touch the line.

If P is any point of the line, the first polar curves of P form a pencil of $(n-1)$-ics. As P travels along the line the base-points of this pencil of $(n-1)$-ics trace out the same locus.

[If $u + kv = 0$ is the pencil and $z = 0$ the given line, the locus is

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}.]$$

Ex. 19. $3(n-1)^2$ of a given pencil of n-ics have a node in general.

The polar line of such a node is the same for each n-ic of the pencil; and the node lies on the $2(n-1)$-ic of Ex. 18.

[The nodes are the intersections of

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial v}{\partial x}\frac{\partial u}{\partial z}$$

other than the intersections of

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} = 0.$$

The case $n = 2$ is familiar. For the case $n = 3$ see Mohrmann, *Math. Annalen*, lxxiv (1913), p. 319.]

§ 4. Harmonic Polar of an Inflexion on a Cubic.

In § 3 suppose the curve is a cubic, and consider the first polar curve, i. e. the polar conic, of any point O.

If $OPQ_1Q_2Q_3$ is a line through O such that

$$OQ_1 . PQ_2 . PQ_3 + OQ_2 . PQ_3 . PQ_1 + OQ_3 . PQ_1 . PQ_2 = 0,$$

P lies on the polar conic of O (§ 1). If O lies on the curve, we may suppose that one of Q_1, Q_2, Q_3, say Q_3, coincides with O, and then

$$OQ_1 . PQ_2 + OQ_2 . PQ_1 = 0$$

or (OP, Q_1Q_2) is an harmonic range. Hence:

If through any point O of a cubic any line is drawn cutting the curve in two other points, the harmonic conjugate of O with respect to these two points lies on the polar conic of O.

If O is an inflexion of the cubic, the tangent at O meets the curve at three consecutive points, which may be considered as forming an harmonic range with any point on the tangent.

Hence the polar conic degenerates into the tangent at O and a straight line called the *harmonic polar* of the inflexion O.* Therefore:

If through an inflexion O of a cubic any line is drawn cutting the curve in two other points, the harmonic conjugate of O with respect to these two points lies on a line through the points of contact of the three tangents from O.

Suppose any line meets the cubic in D_1, D_2, D_3. Let the lines OD_1, OD_2, OD_3 meet the cubic again in E_1, E_2, E_3 and meet the harmonic polar of O in F_1, F_2, F_3. Since the ranges (OF_1, D_1E_1), (OF_2, D_2E_2), (OF_3, D_3E_3) are harmonic, and D_1, D_2, D_3 and F_1, F_2, F_3 are collinear, so are E_1, E_2, E_3. If the points D_1, D_2, D_3 are consecutive on the cubic, so are E_1, E_2, E_3; and both trios of points coincide at inflexions. Hence we have the important theorem:

The line joining two inflexions of a cubic passes through a hird inflexion.

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* See § 7 for the result that the polar conic of an inflexion for any curve egenerates.

Ex. 1. Show that the harmonic polar of $(0, -1, 1)$ for

$$x^3+y^3+z^3+6mxyz = 0$$

is the same for all values of m.

[The polar conic of $(0, -1, 1)$ is $(2mx-y-z)(y-z) = 0$. $2mx-y-z = 0$ is the tangent at the point, and $y-z = 0$ the harmonic polar.]

Ex. 2. Find the harmonic polar of

(i) $(0, 0, 1)$ for $z^2(y-x) = (y+x)(ax^2+by^2)$.

(ii) $(0, -1)$ for $3x^3 = x^2y+y^2+x+y$.

(iii) $(-1, 1, 0)$ for $(x+y+z)^3+6kxyz = 0$.

[(i) $z = 0$, (ii) $x+y = 1$, (iii) $x = y$.]

Ex. 3. Any two chords OPQ, $OP'Q'$ are drawn through the inflexion O of a cubic. Show that PP' and QQ' meet on the harmonic polar of O. Show also that the tangents at P and Q meet on the harmonic polar of O.

[Use the harmonic property of the diagonals of a quadrilateral. Then make the two chords adjacent.]

Ex. 4. Two chords through a point O of a cubic meet the curve again in P and Q, P' and Q'; while they meet the polar conic of O in R and R'. Prove PP', QQ', RR' concurrent.

Show also that the tangents at P, Q, R are concurrent.

[As in Ex. 3.]

Ex. 5. A conic touches a cubic at O and cuts it at P, Q, R, S. Show that OP, OQ, OR, OS meet the cubic again at four points on a conic also touching the cubic at O.

Ex. 6. Any line through an $(n-2)$-ple point O of an n-ic cuts the curve again in Q_1 and Q_2. Find the locus of P, if $(OP, Q_1 Q_2)$ is harmonic.

[The first polar of O less the inflexional tangents at O, if any.]

§ 5. First Polar Curve.

We consider now the intersections with a given curve of the first polar curve of O with respect to this curve. Take $C(0, 0, 1)$ as such a point of intersection, O as $(1, 0, 0)$, and any point as $B(0, 1, 0)$. Take the curve as (i) in § 3. The first polar curve of $(1, 0, 0)$ is

$$0 = b_0 z^{n-1} + 2(c_0 x + c_1 y)z^{n-2} + 3(d_0 x^2 + 2d_1 xy + d_2 y^2)z^{n-3} + \dots.$$

Since this meets the given curve at $(0, 0, 1)$, we must have one of various alternatives.

Firstly, we may have $a = b_0 = 0$.

Then the tangent at $(0, 0, 1)$ to the given curve is $y = 0$ and passes through O. Hence C is the point of contact of a tangent from O. The curve and the first polar curve have C as an ordinary point and the tangents to the curves at C are distinct. Hence a single intersection of the curve and its first polar is at C.

Secondly, we may have $a = b_0 = b_1 = 0$.

The given curve has a node at C, and the first polar curve has C as an ordinary point, the tangent at C to the first polar curve not coinciding with either tangent to the given curve at C. Hence two of the intersections of the curve and its first polar coincide at C.*

Thirdly, the given curve may have a cusp at C. Taking $x = 0$ as the tangent to the cusp, we have

$$a = b_0 = b_1 = c_1 = c_2 = 0.$$

We see that the first polar touches the given curve at the cusp, and three of the intersections of the curves coincide at C.*

We sum up our results thus:

The first polar curve of O meets the given curve once at the point of contact of each tangent from O to the given curve, twice at each node, and thrice at each cusp of the given curve.

It follows at once that, if m is the class of the given n-ic (Ch. IV, § 2), δ the number of nodes, and κ the number of cusps,

$$m = n(n-1) - 2\delta - 3\kappa \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

For m is the number of tangents from O, i.e. the number of intersections of the first polar of O with the given curve at points which are not multiple points of the given curve.

If the curve has τ bitangents and ι inflexions, the reciprocal curve is of degree m, is of class n, and has τ nodes and ι cusps (Ch. IV, § 7).

Hence $$n = m(m-1) - 2\tau - 3\iota \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

If O lies on a curve, the first polar touches the curve at O, i.e. meets it twice at O (§ 3). Hence two of the tangents from a point O on a curve must be considered as coinciding with the tangent at O.

Ex. 1. Find the tangents from $(1, 1, 1)$ to $(x^2+y^2)z = 2x^3$.

[The first polar curve of $(1, 1, 1)$ is $y^2-5x^2+2z(x+y) = 0$. This meets the curve where $(y^2-2xy+x^2)(y^2+2xy-x^2) = 0$. The points of contact of the tangents are therefore $(1, -1 \pm 2^{\frac{1}{2}}, 1 \pm 2^{-\frac{1}{2}})$.]

Ex. 2. Find the tangents:

(i) From $(1, 1, 1)$ to $x^3+y^3 = 2z^3$.
(ii) From $(0, 1, -1)$ to $x^3+y^3+z^3 = 5xyz$.
(iii) From $(11, 16, 9)$ to $x^3+y^3 = 3xyz$.
(iv) From $(\frac{7}{3}a, 3a)$ to $ay^2 = x^3$.
(v) From (a, a) to $x^3+y^3 = a^3$.
(vi) From $(0, 0, 1)$ to $ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2) = 0$.

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* See Ch. I, § 7 or Ch. VI, § 2.

[The points of contact are given by

(i) $x^4+2x^3y+2xy^3+y^4=0$.
(ii) $x/y=2$ or $-1\pm\sqrt{2}$.
(iii) $(x-y)(2x-y)(8x^2+xy+11y^2)=0$.
(iv) $x=at^2$, $y=at^3$; where $t=1, 2, -3$.
(v) $x^2y^2(3x^2-2xy+3y^2)=0$.
(vi) $c^2x=\{a\pm(a^2-c^2)^{\frac{1}{2}}\}\{b\pm(b^2-c^2)^{\frac{1}{2}}\}y$.]

Ex. 3. Extend the result of Ch. I, § 9, Ex. 6 to the case in which tangents are drawn to an n-ic from a $(n-3)$-ple point.

[The tangents from (0, 0, 1) to

$$z^3u_{n-3}+3z^2u_{n-2}+3zu_{n-1}+u_n=0$$

touch at points on the first polar

$$(zu_{n-3}+u_{n-2})^2+u_{n-3}u_{n-1}-u^2_{n-2}=0$$

and meet the curve again on

$$(zu_{n-3}+u_{n-2})^2+4(u_{n-3}u_{n-1}-u^2_{n-2})=0.]$$

§ 6. Equation of Hessian.

In § 1 we defined the Hessian of a curve as the locus of points whose polar conic with respect to the given curve degenerates into a line-pair. The polar conic of (X, Y, Z) with respect to $f(x, y, z)=0$ is

$$x^2\frac{\partial^2 f}{\partial X^2}+y^2\frac{\partial^2 f}{\partial Y^2}+z^2\frac{\partial^2 f}{\partial Z^2}+2yz\frac{\partial^2 f}{\partial Y\partial Z}+2zx\frac{\partial^2 f}{\partial Z\partial X}+2xy\frac{\partial^2 f}{\partial X\partial Y}=0.$$

This is a line-pair if (X, Y, Z) lies on

$$H\equiv\begin{vmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y} & \dfrac{\partial^2 f}{\partial x\partial z}\\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y\partial z}\\ \dfrac{\partial^2 f}{\partial z\partial x} & \dfrac{\partial^2 f}{\partial z\partial y} & \dfrac{\partial^2 f}{\partial z^2}\end{vmatrix}=0,$$

which is the equation of the Hessian of $f=0$. If f is of degree n in x, y, z, each element of the determinant is of degree $n-2$. Hence

The Hessian of an n-ic is a $3(n-2)$-ic.

To find the Hessian of a curve whose equation is given in Cartesian coordinates we make the equation homogeneous by writing x/z and y/z for x and y. Then after finding the Hessian we put $z=1$.

Multiply the columns of the determinant H by x, y, z, and add the first two columns to the third; then multiply the rows by x, y, z and add the first two rows to the third. We shall obtain the identity *

$$z^2 H \equiv (n-1)^2 \left\{ 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial^2 f}{\partial y^2} \right\}$$

$$+ n(n-1) \left\{ \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \right\} f \quad . \quad . \quad . \quad \text{(i)}.$$

This gives an alternative method of calculating the equation of the Hessian which is convenient when the equation of the curve is given in Cartesian coordinates, or when an approximation to the Hessian is required in the neighbourhood of $(0, 0, 1)$.

Ex. Find the Hessian of a curve $f(r, \theta) = 0$ given in polar coordinates.

[Putting $z = 1$ in § 6 (1) the expression in the first brackets $\{\}$ is

$$\frac{1}{r^2}\left\{ 2\frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta}\frac{\partial^2 f}{\partial r \partial \theta} - \left(\frac{\partial f}{\partial \theta}\right)^2 \frac{\partial^2 f}{\partial r^2} - \left(\frac{\partial f}{\partial r}\right)^2 \frac{\partial^2 f}{\partial \theta^2} \right\}$$

$$- \frac{1}{r^3}\frac{\partial f}{\partial r}\left\{ r^2 \left(\frac{\partial f}{\partial r}\right)^2 + 2\left(\frac{\partial f}{\partial \theta}\right)^2 \right\},$$

and the expression in the second brackets $\{\}$ is

$$\frac{1}{r^2}\frac{\partial^2 f}{\partial r^2}\left(\frac{\partial^2 f}{\partial \theta^2} + r\frac{\partial f}{\partial r}\right) - \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2 .]$$

§ 7. Intersections of a Curve with its Hessian.

By relation (i) of § 6 the intersections of an n-ic $f = 0$ with its Hessian $H = 0$ are the same as the intersections of $f = 0$ with

$$K \equiv 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} - \left(\frac{\partial f}{\partial y}\right)^2 \frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2 \frac{\partial^2 f}{\partial y^2} = 0;$$

except that $f = 0$ and $K = 0$ meet, not only at the intersections of $f = 0$ and $H = 0$, but also twice at each intersection of $f = 0$ and $z = 0$.

To determine the intersections of $f = 0$ and $H = 0$ which coincide with any point O on $f = 0$, we shall take O as $(0, 0, 1)$; and for this purpose we may evidently replace the Hessian $H = 0$ by $K = 0$.

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* For by Euler's theorem on homogeneous functions

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} \equiv nf, \quad x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} + z\frac{\partial^2 f}{\partial x \partial z} \equiv (n-1)\frac{\partial f}{\partial x}, \text{ \&c.}$$

Suppose, then, the triangle of reference chosen so that any point O on $f = 0$ is $(0, 0, 1)$ and the tangent at O is $y = 0$. We have

$$f \equiv byz^{n-1} + (c_0x^2 + 2c_1xy + c_2y^2)z^{n-2} + (d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) + \dots.$$

The polar conic of O is

$$c_0x^2 + c_2y^2 + (n-1)byz + 2c_1xy = 0.$$

If this is a line-pair, either $c_0 = 0$ and the coefficient of z^{n-1} in f is a factor of the coefficient of z^{n-2}, i. e. the curve $f = 0$ has an inflexion at O, or else $b = 0$ and O is a multiple point of the curve. Hence

The Hessian meets a curve only at the inflexions and multiple points of the curve.

In the case $b \neq 0$, $c_0 = 0$

$$K \equiv \{(8bc_1^2 - 6b^2d_1)y - 6b^2d_0x\}z^{3n-5} + \dots,$$

keeping only the highest power of z.

Hence the curve and $K = 0$ (and therefore the curve and the Hessian) meet only once at O.

Suppose now $f = 0$ has a node at O, so that $b = 0$.

In this case

$$K \equiv 8(c_1^2 - c_0c_2)(c_0x^2 + 2c_1xy + c_2y^2)z^{3n-6} + \dots.$$

Hence $K = 0$ (and $H = 0$) has a node at O with the same tangents as the given curve. The curve and Hessian therefore meet six times at the node (Ch. VI, § 2, Ex. (iii)).

Suppose now that O is a cusp of the given curve at which $y = 0$ is the tangent. Then

$$f \equiv y^2z^{n-2} + (d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3)z^{n-3} + \dots,$$

and

$$K \equiv -24(d_0x + d_1y)y^2z^{3n-7} + \dots.$$

Hence $K = 0$ (and $H = 0$) has a triple point at O at which two tangents coincide with the cuspidal tangent of the given curve.

The curve and the Hessian intersect eight times at the cusp (Ch. VI, § 2, Ex. (iv)).

Summing up the above results we have:

The intersections of a curve with its Hessian lie one at each inflexion, six at each node, and eight at each cusp of the given curve.

If δ is the number of nodes, κ of cusps, and ι of inflexions of the n-ic,

$$\iota = 3n(n-2) - 6\delta - 8\kappa.$$

For the curve and Hessian meet altogether in $3n(n-2)$ points.

The number of intersections of the Hessian with the curve at the nodes and cusps may also be obtained in the following manner (due to Cayley). The number is evidently independent of n, so that there are constants A and B such that

$$\iota = 3n(n-2) - A\delta - B\kappa, \quad \kappa = 3m(m-2) - A\tau - B\iota;$$

the latter equation being derived from the reciprocal curve as in § 5.

But by § 5

$$m = n(n-1) - 2\delta - 3\kappa, \quad n = m(m-1) - 2\tau - 3\iota.$$

Eliminating m, τ, ι from these four equations, we have

$$(A-6)\{n^2-2n-2\delta-3\kappa)(n^2-2\delta-3\kappa)+4\delta+6\kappa\}$$
$$+(3A-2B-2)\{-3n^2+6n+A\delta+(B-1)\kappa\} = 0.$$

Since this must hold for all values of n,

$$A-6 = 3A-2B-2 = 0, \text{ or } A = 6, B = 8$$

Ex. 1. Find the Hessian of $x^3+y^3+z^3+6mxyz = 0$, and find its inflexions.

[The Hessian is $m^2(x^3+y^3+z^3) = (2m^3+1)xyz$, meeting the original curve where $xyz = 0$. Hence the inflexions are the intersections of the curve with the sides of the triangle of reference.]

Ex. 2. Find the Hessians of the following curves:

(i) $y^p z^q = x^{p+q}$.

(ii) $(x+y+z)^3+6kxyz = 0$.

(iii) $x^3+y^3+z^3 = h(x+y+z)^3$.

(iv) $x^3 = z(x^2 \pm y^2)$.

(v) $x(x^2-3y^2)+a(x^2+y^2) = 0$.

(vi) $6z(x^2+y^2) = (x+y)(x^2-4xy+y^2)$.

(vii) $x^3+y^3 = xy(x+y+z)$.

(viii) $z^2x = y(y-x)(y-k^2x)$.

(ix) $y^2z = 4x^3-g_2xz^2-g_3z^3$.

(x) $a_0x^3+3a_1x^2y+3a_2xy^2+a_3y^3+6xyz = 0$.

(xi) $y^2z+x^2y+z^2x+6pxyz = 0$.

(xii) $ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2) = 0$.

(xiii) $x^3y+y^3z+z^3x = 0$.

(xiv) $(yz+x^2)^2 = xy^3$.

[(i) $x^{p+q-2}y^{2p-2}z^{2q-2} = 0$.

(ii) $(x+y+z)(2yz+2zx+2xy-x^2-y^2-z^2)+2kxyz = 0$. The inflexions lie on $(x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z) = 0$.

(iii) $xyz = h(x+y+z)(yz+zx+xy)$.

(iv) $3xy^2 = z(y^2 \pm x^2)$.

(v) $3x(x^2-3y^2)+a(x^2+y^2) = 0$.

(vi) $2z(x^2+y^2)+(x+y)(x^2-4xy+y^2)=0$.

(vii) $3(x^3+y^3)=xy(z-x-y)$.

(viii) $z^2\{(1+k^2)x-3y\}+x\{k^4x^2-k^2(1+k^2)xy+(1-k^2+k^4)y^2\}=0$.

(ix) $12xy^2=z(12g_2x^2+36g_3xy+g_2^2z^2)$.

(x) $a_0x^3-a_1x^2y-a_2xy^2+a_3y^3=2xyz$.

(xi) $x^3+y^3+z^3-9p^2(xz^2+yx^2+zy^2)-3(18p^3+1)xyz=0$.

(xii) $(b^2-c^2)(by+cz)x^2+\ldots+\ldots$

$$=\{x^2+y^2+z^2\}\{a(b^2-c^2)x+\ldots+\ldots\}.$$

(xiii) $x^5z+y^5x+z^5y=5x^2y^2z^2$. Similarly for $x^ny+y^nz+z^nx=0$.

(xiv) $8(yz+x^2)^3+16xy^3(yz+x^2)+3y^6=0$.]

Ex. 3. The number of inflexions of an n-ic is even or odd according as the n is even or odd.

A curve of odd degree has at least one real inflexion.

Ex. 4. If a curve has r-point contact with its tangent at O, the Hessian has $(r-2)$-point contact with the same tangent at O; and the curve and Hessian meet $r-2$ times at O.

[The Hessian of $yu+v=0$, where the terms of lowest degree in v are of the r-th degree, is of the form $yU+V=0$, where the terms of lowest degree in V are of the $(r-2)$-th degree.]

Ex. 5. The Hessian of the 'triangular-symmetric' curve

$$(x/a)^n+(y/b)^n+(z/c)^n=0$$

is the sides of the triangle of reference $n-2$ times.

Each side passes through n points of the curve at which the tangent has n-point contact and passes through the opposite vertex.

[By Ex. 4 each such point counts as $(n-2)$ inflexions.]

Ex. 6. The locus of the inflexions of $zx^2y=y^3(2x+y)+ax^4$ for varying values of a is $2y^3+x^2z=0$.

[The Hessian is $x^2(2ax^4+4xy^3+x^2yz+8y^4)=0$. Now eliminate a.]

Ex. 7. The locus of the inflexions of a pencil of n-ics is a $6(n-1)$-ic. Explain the case $n=2$.

Ex. 8. The locus of the points whose polar conic with respect to a given n-ic touches a given line is a $2(n-2)$-ic.

This $(n-2)$-ic divides the plane into parts in one of which the n-ic has no real inflexion or crunode, while in the other the n-ic has no acnode.

[If $f=0$ is the n-ic and $z=0$ the given line, the $2(n-1)$-ic is

$$F\equiv\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2=0.$$

If $F>0$, the degenerate polar conic of a point on the Hessian is unreal.

In the case $n=3$, the $2(n-2)$-ic is called the 'polo-conic' of the given line.]

Ex. 9. The $2(n-2)$-ic of Ex. 8 is the envelope of the $(n-2)$-th polar curve of any point on the given line.

Ex. 10. If the given line of Ex. 8 is the tangent at an inflexion O, the locus has a node at O with the given line as one tangent at O.

Ex. 11. The tangents to a curve at the multiple point (0, 0, 1) are $u = 0$. Show that the tangents to the Hessian at the same point are in general $u = 0$ and
$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2.$$

Ex. 12. A k-ple point O of a curve is in general a $(3k-4)$-ple point of the Hessian, and the tangents to the curve at O are tangents to the Hessian at O.

Ex. 13. The lines joining (0, 0, 1) to the inflexions of $zx^py^q + u = 0$, where u is homogeneous of degree $p+q+1$ in x and y, are
$$q(q+1)x^2\frac{\partial^2 u}{\partial x^2} + p(p+1)y^2\frac{\partial^2 u}{\partial y^2} = 2pqxy\frac{\partial^2 u}{\partial x \partial y}.$$
If $p = 0$, they become $\dfrac{\partial^2 u}{\partial x^2} = 0$. If $p = q$, they become
$$p(p+1)u = xy\frac{\partial^2 u}{\partial x \partial y}.$$

[Eliminate z between the equations of curve and Hessian; or use Ex. 14 (v).]

Ex. 14. Suppose that λ, μ, ν are homogeneous functions of x and y of degrees p, q, r not less than 2. With the notation
$$2a_{\lambda\mu} \equiv \lambda_{11}\mu_{22} + \lambda_{22}\mu_{11} - 2\lambda_{12}\mu_{12},$$
$$2b_{\lambda\mu,\nu} \equiv \begin{vmatrix} \lambda_{11} & \mu_{12} & \nu_1 \\ \lambda_{12} & \mu_{22} & \nu_2 \\ \lambda_1 & \mu_2 & 0 \end{vmatrix} + \begin{vmatrix} \mu_{11} & \lambda_{12} & \nu_1 \\ \mu_{12} & \lambda_{22} & \nu_2 \\ \mu_1 & \lambda_2 & 0 \end{vmatrix},$$
$$c_{\lambda\mu\nu} \equiv n(\lambda a_{\mu\nu} + \mu a_{\nu\lambda} + \nu a_{\lambda\mu}) + (n-1)(b_{\mu\nu,\lambda} + b_{\nu\lambda,\mu} + b_{\lambda\mu,\nu}),$$
where the suffixes 1 and 2 denote partial differentiation with respect to x and y,

(i) Prove that
$$a_{\lambda\mu} = a_{\mu\lambda}, \qquad b_{\lambda\mu,\nu} = b_{\mu\lambda,\nu}$$
and that
$$2(p-1)(q-1)(r-1)b_{\lambda\mu,\nu} \equiv -r(r-1)(p+q-2)\nu a_{\lambda\mu}$$
$$+ p(p-1)(p-q)\lambda a_{\mu\nu} + q(q-1)(q-p)\mu a_{\lambda\nu},$$
$$(p-1)(q-1)(r-1)c_{\lambda\mu\nu}$$
$$\equiv (p-1)\{n(p-q)(p-r) + (n-p)(p-q-r+1)\}\lambda a_{\mu\nu}$$
$$+ (q-1)\{n(q-r)(q-p) + (n-q)(-p+q-r+1)\}\mu a_{\nu\lambda}$$
$$+ (r-1)\{n(r-p)(r-q) + (n-r)(-p-q+r+1)\}\nu a_{\lambda\mu}.$$

(ii) Show that the Hessian of
$$z^{n-k}u + z^{n-k-1}v + z^{n-k-2}w + z^{n-k-3}t + z^{n-k-4}s + \ldots = 0,$$
where $k \geqslant 2$, and $u, v, w, t, s, \ldots$ are homogeneous of degree
$$k, k+1, k+2, k+3, k+4, \ldots$$
in x and y, is
$$0 = z^{3n-3k-2}\tfrac{1}{3}c_{uuu} + z^{3n-3k-3}c_{uuv} + z^{3n-3k-4}(c_{uvv} + c_{uuw})$$
$$+ z^{3n-3k-5}(\tfrac{1}{3}c_{vvv} + 2c_{uvw} + c_{uut}) + z^{3n-3k-6}(c_{uww} + c_{vvw} + 2c_{uvt} + c_{uus}) + \ldots.$$

(iii) Hence show that the Hessian when expressed in terms of $z, u, v, w, \dots, a_{uu}, a_{uv}, \dots,$ is

$$0 = -\frac{z^{3n-3k-2}}{k-1}(n-k)\,ua_{uu} + \frac{z^{3n-3k-3}}{(k-1)^2}\{-2(k-1)(n-k)\,ua_{uv} + [n(-k+3)+(k+1)(k-2)]\,va_{uu}\}$$

$$+\frac{z^{3n-3k-4}}{k(k-1)^2}\{-(k-1)^2(n-k-1)(ua_{uu}+2va_{uu}) - 2k(k-1)(n-k)\,ua_{uw} + k[n(-k+7)+(k+2)(k-3)]\,wa_{uu}\} + \dots.$$

(iv) In particular the Hessian of $zu+v=0$, where u and v are of degrees $(n-1)$ and n in x and y, is

$$(n-2)\,va_{uu}z = uva_{uu} - 2(n-2)\,ua_{uu}.$$

(v) The inflexions of $zu+v=0$ lie on

$$(n-1)\,a_{uu}z + (n-2)\,a_{uv} = 0 \quad \text{and on} \quad (n-1)\,va_{uu} = (n-2)\,ua_{uu}.$$

[(i) In the definition of $b_{\lambda\mu,\nu}$ substitute for $\lambda_1\nu_1$, &c., from the identities

$$2(p-1)(q-1)\lambda_1\mu_1 \equiv q(q-1)\mu\lambda_{11} + p(p-1)\lambda\mu_{11} - 2y^2a_{\lambda\mu},$$
$$2(p-1)(q-1)\lambda_2\mu_2 \equiv q(q-1)\mu\lambda_{22} + p(p-1)\lambda\mu_{22} - 2x^2a_{\lambda\mu},$$
$$(p-1)(q-1)(\lambda_1\mu_2+\lambda_2\mu_1) \equiv q(q-1)\mu\lambda_{12} + p(p-1)\lambda\mu_{12} + 2xya_{\lambda\mu};$$

which are derived from $(p-1)\lambda_1 \equiv x\lambda_{11} + y\lambda_{12}$, &c.

(v) Eliminate v between the equations of the curve and Hessian.]

Ex. 15. The ratio of the curvatures of the branch of an n-ic through a k-ple point O and of the branch of the Hessian touching this branch at O is
$$(n-k)(k-1) : (nk-3n-k^2+k+2).$$

[If the curve is
$$0 = y(x^{k-1} + ax^{k-2}y + \dots) + bx^{k+1} + \dots,$$
by Ex. 14 (iii) the branch of the Hessian touching $y=0$ at the origin is
$$0 = y(n-k)(k-1) + (nk-3n-k^2+k+2)bx^2 + \dots,$$
using Newton's diagram.]

Ex. 16. The curvatures of two branches of an n-ic and its Hessian which touch at a crunode of the n-ic are in general in opposite directions.

[If $k=2$ in Ex. 15, the ratio of the curvatures is $(2-n) : n$.]

Ex. 17. In what cases is it true that each branch of the Hessian touching a given n-ic at a k-ple point is inflexional?

[$nk-3n-k^2+k+2=0$ gives $n=10$, $k=4$; $n=10$, $k=7$; or $n=9$, $k=5$.]

§ 8. The Steinerian.

Suppose that the first polar curve of a point $P(\xi, \eta, \zeta)$ with respect to a curve $f(x, y, z) = 0$ of degree n has a double point at $Q(X, Y, Z)$. The locus of P is called the *Steinerian* of the given curve. Writing down the conditions that the first polar curve of P, namely,

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}=0,$$

should have a double point at Q, we have by Ch. II, § 4,

$$\xi\frac{\partial^2 f}{\partial X^2}+\eta\frac{\partial^2 f}{\partial X\partial Y}+\zeta\frac{\partial^2 f}{\partial X\partial Z}=0,\quad \xi\frac{\partial^2 f}{\partial Y\partial X}+\eta\frac{\partial^2 f}{\partial Y^2}+\zeta\frac{\partial^2 f}{\partial Y\partial Z}=0,$$
$$\xi\frac{\partial^2 f}{\partial Z\partial X}+\eta\frac{\partial^2 f}{\partial Z\partial Y}+\zeta\frac{\partial^2 f}{\partial Z^2}=0\quad . \quad . \quad . \quad . \quad \text{(i)}.$$

Eliminating X, Y, Z from the equations (i) and replacing ξ, η, ζ by x, y, z we have the equation of the Steinerian. It may be shown that, since each of the three equations (i) is of degree $n-2$ in X, Y, Z, the Steinerian is in general of degree $3(n-2)^2$.*

Eliminating ξ, η, ζ from equations (i) we have

$$\begin{vmatrix}\frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial X\partial Y} & \frac{\partial^2 f}{\partial X\partial Z}\\ \frac{\partial^2 f}{\partial Y\partial X} & \frac{\partial^2 f}{\partial Y^2} & \frac{\partial^2 f}{\partial Y\partial Z}\\ \frac{\partial^2 f}{\partial Z\partial X} & \frac{\partial^2 f}{\partial Z\partial Y} & \frac{\partial^2 f}{\partial Z^2}\end{vmatrix}=0\quad . \quad . \quad . \quad \text{(ii)}.$$

Hence:

The locus of the double points of the first polar curves of a given curve is its Hessian.

The conditions that P should be a node of the polar conic of Q

$$x^2\frac{\partial^2 f}{\partial X^2}+y^2\frac{\partial^2 f}{\partial Y^2}+z^2\frac{\partial^2 f}{\partial Z^2}$$
$$+2yz\frac{\partial^2 f}{\partial Y\partial Z}+2zx\frac{\partial^2 f}{\partial Z\partial X}+2xy\frac{\partial^2 f}{\partial X\partial Y}=0$$

are the equations (i).

Hence:

If the first polar curve of P has a node at Q, the polar conic of Q is a line-pair meeting at P.

Suppose now that corresponding points P' and Q' on the

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* Salmon's *Higher Algebra*, § 76. Another proof is given in § 9, Ex. 9. For an extension of the theorems in §§ 8, 9, see Henrici, *Proc. London Math. Soc.*, ii (1868), pp. 112, 183.

Steinerian and Hessian consecutive to P and Q have coordinates

$(\xi+d\xi,\ \eta+d\eta,\ \zeta+d\zeta)$ and $(X+dX,\ Y+dY,\ Z+dZ)$

respectively. Then, since the first polar curves of P and P' go through Q and Q' respectively,

$$\xi\frac{\partial f}{\partial X}+\eta\frac{\partial f}{\partial Y}+\zeta\frac{\partial f}{\partial Z}=0,$$

$$(\xi+d\xi)\left(\frac{\partial f}{\partial X}+\frac{\partial^2 f}{\partial X^2}dX+\frac{\partial^2 f}{\partial X\partial Y}dY+\frac{\partial^2 f}{\partial X\partial Z}dZ\right)+\ldots+\ldots=0;$$

or, using (i),

$$\left.\begin{aligned}\xi\frac{\partial f}{\partial X}+\eta\frac{\partial f}{\partial Y}+\zeta\frac{\partial f}{\partial Z}&=0\\ d\xi\frac{\partial f}{\partial X}+d\eta\frac{\partial f}{\partial Y}+d\zeta\frac{\partial f}{\partial Z}&=0\end{aligned}\right\}$$

These equations show that the polar line of Q

$$x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}+z\frac{\partial f}{\partial Z}=0$$

is identical with the line through the points P and P', which is the tangent at P to the Steinerian in the limit. Hence:

If the first polar curve of P has a node at Q, the polar line of Q is the tangent at P to the locus of P.

A tangent from (x', y', z') to the Steinerian is the polar line of a point (X, Y, Z) on the Hessian, such that (ii) holds and also

$$x'\frac{\partial f}{\partial X}+y'\frac{\partial f}{\partial Y}+z'\frac{\partial f}{\partial Z}=0.$$

Therefore the tangents from (x', y', z') to the Steinerian are the polar lines of the intersections of the Hessian with the first polar curve of (x', y', z'), other than the double points of the original curve. This Hessian and first polar curve are of degrees $3(n-2)$, $n-1$, and may be shown to meet twice at each node and four times at each cusp of the original curve (Ch. VI, § 2). Hence:

The class of the Steinerian is

$$3(n-1)(n-2)-2\delta-4\kappa,$$

where δ, κ are the number of nodes and cusps of the given curve.[*]

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[*] For the effect of multiple points of a curve on the degree and class of the Steinerian, see Koehler, *Bull. de la Soc. Math. de France*. i (1873), pp. 124-9.

§ 9. The Cayleyan.

If the first polar curve of P has a double point at Q, the envelope of PQ is called the *Cayleyan* (or 'Pippian') of the given curve.

Suppose that the line $\lambda x + \mu y + \nu z = 0$ is the polar line of (X, Y, Z) with respect to $f(x, y, z) = 0$. Then

$$\frac{\partial f}{\partial X}\Big/\lambda = \frac{\partial f}{\partial Y}\Big/\mu = \frac{\partial f}{\partial Z}\Big/\nu.$$

Therefore the given line is the polar line of each of the $(n-1)^2$ intersections of

$$\lambda\frac{\partial f}{\partial z} = \nu\frac{\partial f}{\partial x} \quad\text{and}\quad \mu\frac{\partial f}{\partial z} = \nu\frac{\partial f}{\partial y} \quad . \quad . \quad . \quad . \quad (\text{i}),$$

which are the first polar curves of $(\nu, 0, -\lambda)$ and $(0, \nu, -\mu)$.

Hence:

In general a given line has $(n-1)^2$ poles with respect to a given n-ic.

Suppose now that two poles of $\lambda x + \mu y + \nu z = 0$ coincide at the point (X, Y, Z). The two curves (i) touch at this point. The tangents at the point to the curves (i) are

$$x\left(\lambda\frac{\partial^2 f}{\partial Z\,\partial X} - \nu\frac{\partial^2 f}{\partial X^2}\right) + y\left(\lambda\frac{\partial^2 f}{\partial Z\,\partial Y} - \nu\frac{\partial^2 f}{\partial X\,\partial Y}\right) + z\left(\lambda\frac{\partial^2 f}{\partial Z^2} - \nu\frac{\partial^2 f}{\partial X\,\partial Z}\right) = 0,$$

$$x\left(\mu\frac{\partial^2 f}{\partial Z\,\partial X} - \nu\frac{\partial^2 f}{\partial Y\,\partial X}\right) + y\left(\mu\frac{\partial^2 f}{\partial Z\,\partial Y} - \nu\frac{\partial^2 f}{\partial Y^2}\right) + z\left(\mu\frac{\partial^2 f}{\partial Z^2} - \nu\frac{\partial^2 f}{\partial Y\,\partial Z}\right) = 0.$$

Identifying them and eliminating λ, μ, ν between the relations thus obtained we get equation (ii) of § 8. Hence (X, Y, Z) lies on the Hessian. Moreover, the corresponding point (ξ, η, ζ) of the Steinerian lies on either of the (identical) tangents by equations (i) of § 8. Summing up, we have:

If two poles of a line coincide, the coincident poles lie on the Hessian and the line touches the Steinerian at the corresponding point. The Cayleyan is the envelope of the tangent at the point of contact of two first polar curves of the given curve.

Ex. 1. If the first polar curve of $P(1, 0, 0)$ with respect to $f = 0$ has a node at Q $(0, 0, 1)$,

$$f \equiv az^n + bz^{n-1}y + cz^{n-2}y^2 + z^{n-3}(d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) + \dots.$$

Ex. 2. Use Ex. 1 to show that Q lies on the Hessian and that the polar conic of Q is a line-pair meeting at P.

Ex. 3. The second polar of P touches the Hessian at Q.

[The common tangent is $d_0x + d_1y = 0$.]

Ex. 4. The tangent at Q to the Hessian is the harmonic conjugate of PQ for the tangents to the first polar of P at Q.

[The tangents are $d_0x^2 + 2d_1xy + d_2y^2 = 0$.]

Ex. 5. The tangent at P to the Steinerian is the harmonic conjugate of PQ for the degenerate polar conic of Q.

[The tangent at P is the polar line of Q, which is $naz + by = 0$. The polar conic of Q is

$$n(n-1)az^2 + 2(n-1)bzy + 2cy^2 = 0.]$$

Ex. 6. The first polar of any point on the tangent at P to the Steinerian touches PQ at Q.

Ex. 7. If PQ touches the Hessian at Q, it touches the Cayleyan at Q.

[If P and Q, P' and Q' are consecutive pairs of points on Steinerian and Hessian, PQ and $P'Q'$ meet at Q'.]

Ex. 8. The Steinerian and Cayleyan touch the inflexional tangents of the given curve.

[These tangents are the polar lines of the inflexions.]

Ex. 9. The locus of § 2, Ex. 12 (ii) meets the Hessian at the points corresponding to the intersections of the Steinerian with the given conic.

Ex. 10. The Hessian and Steinerian of the 'triangular-symmetric' curve

$$(x/a)^n + (y/b)^n + (z/c)^n = 0$$

degenerate into the sides of the triangle of reference, and the Cayleyan into the vertices of this triangle.

Ex. 11. The polar cubic of O with respect to a quartic degenerates into a line and a conic. Show that O is a node of the Steinerian and that the line is a 4-ple tangent of the Cayleyan.

[Take O as $(0, 0, 1)$ and the quartic as

$$z^4 + z^3u_1 + z^2xy + u_4 = 0.]$$

§ 10. Jacobian of Three Curves.

If $U = 0$, $V = 0$, $W = 0$ are the equations of three curves, the curve

$$J \equiv \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix} = 0,$$

or, as it is usually written,

$$\frac{\partial(U, V, W)}{\partial(x, y, z)} = 0$$

is called the *Jacobian* of the three curves.

The polar lines of (x, y, z) with respect to the three curves are

$$X\frac{\partial U}{\partial x}+Y\frac{\partial U}{\partial y}+Z\frac{\partial U}{\partial z}=0, \text{ \&c.},$$

X, Y, Z being current coordinates.

Hence the Jacobian is the locus of a point whose polar lines with respect to the three given curves are concurrent.

It follows that the Jacobian of three curves projects into the Jacobian of their projections.

If the curves are of degrees n_1, n_2, n_3, the Jacobian is evidently of degree

$$n_1+n_2+n_3-3.$$

We at once verify that, if $U=0$ and $V=0$ are straight lines, the Jacobian is the first polar curve of their intersection with respect to $W=0$. Hence a first polar curve is only a particular case of a Jacobian.

In § 5 we discussed the intersections of a curve with any first polar. We may generalize this investigation by finding the intersections with a given n-ic $W=0$ of the Jacobian of an n_1-ic $U=0$, an n_2-ic $V=0$, and the n-ic $W=0$.

Multiplying the columns of the determinant J by x, y, z and adding the first two columns to the third, we can express Jz in the form

$$Jz\equiv\begin{vmatrix}\frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & n_1U\\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & n_2V\\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & nW\end{vmatrix}.$$

Suppose that $(0, 0, 1)$ is an intersection of $W=0$ and $J=0$. Take the tangent to $W=0$ at $(0, 0, 1)$ as $y=0$. Then

$$W\equiv b_0yz^{n-1}+(c_0x^2+2c_1xy+c_2y^2)z^{n-2}+\dots.$$

Arranging the determinant just given for Jz in descending powers of z, and taking for U and V the most general expressions of degrees n_1 and n_2 also arranged in descending powers of z, we readily find that the Jacobian passes through $(0, 0, 1)$ in the following cases.

(i) That curve of the pencil $U+kV=0$ which passes through $(0, 0, 1)$ touches $W=0$ there.

The Jacobian does not in general touch $W=0$ at $(0, 0, 1)$ in this case. Hence the Jacobian cuts $W=0$ once at each point of contact of $W=0$ with a curve of the pencil

$U+kV=0$. This may also be proved by identifying the tangents to $W=0$ and $U+kV=0$ at any point of intersection.

(ii) $b_0=0$.

Now $W=0$ has a node at $(0, 0, 1)$ and the Jacobian passes through $(0, 0, 1)$. Hence the Jacobian meets $W=0$ twice at every node of $W=0$.

(iii) $b_0=c_0=c_1=0$.

Now $W=0$ has a cusp at $(0, 0, 1)$ with $y=0$ as cuspidal tangent. The Jacobian touches $y=0$ at $(0, 0, 1)$. Hence the Jacobian meets $W=0$ thrice at every cusp of $W=0$.

These results are true only if U and V are perfectly general. Some important special cases are discussed in the examples below.

Ex. 1. If the first polar curves of P with respect to three given curves are concurrent at Q, Q is on the Jacobian.

Ex. 2. Any three curves of the family $aU+bV+cW=0$ have the same Jacobian as the three curves $U=0$, $V=0$, $W=0$, provided the equation of the Jacobian does not vanish identically.

Ex. 3. The Jacobian of $U=0$, $V=0$, $W=0$ is the locus of the nodes of the family $aU+bV+cW=0$, and also of $aVW+bWU+cUV=0$.

Ex. 4. If three curves pass through O, their Jacobian passes through O. If the curves are of the same degree, O is a node of the Jacobian. If two of them are of the same degree, the Jacobian touches the third curve at O.

[In Ex. 4 to 7 take O as $(0, 0, 1)$.]

Ex. 5. If O is a cusp of $W=0$, the Jacobian of $U=0$, $V=0$, $W=0$ touches $W=0$ at O.

If O is a k-ple point of $W=0$, the Jacobian has a $(k-1)$-ple point at O.

Ex. 6. If O is a k_1-ple, k_2-ple, k_3-ple point of an n_1-ic, n_2-ic, n_3-ic respectively, the Jacobian of the curves is an $(n_1+n_2+n_3-3)$-ic with O as a multiple point of order not less than $k_1+k_2+k_3-2$.

If $n_1=n_2$ and $k_1=k_2$, O is a $(2k_1+k_3-2)$-ple point of the Jacobian at which k_3 tangents coincide with those of the n_3-ic.

If $n_1=n_2=n_3$ and $k_1=k_2=k_3$, O is in general a $(3k_1-1)$-ple point of the Jacobian.

Ex. 7. If O is a cusp of a curve, the Jacobian of this curve and any other two curves through O has a node at O one of whose branches touches the given curve at O.

If the two other curves have the same degree, the Jacobian has a cusp at O with the same tangent as the given curve.

Ex. 8. The number of curves of a pencil of N-ics $U+kV=0$ which touch a given curve $W=0$ of degree n and class m is in general $2n(N-1)+m$.

[The points of contact are the $(2N+n-3)n$ intersections of $W=0$ and the Jacobian of $U=0$, $V=0$, $W=0$ less the nodes of $W=0$ counted twice and the cusps counted thrice. Now use equation (i) of § 5.]

Ex. 9. A pencil of N-ics has as its base-points each of the δ nodes and κ cusps of an n-ic $(n > N)$ and other points of the n-ic. Show that the number of N-ics touching the n-ic at a point other than a base-point is

$$(n-N)(N-n+3)+4D-2\,;$$

where $$D = \tfrac{1}{2}(n-1)(n-2)-\delta-\kappa.$$

[By Ex. 4, 6, 7 the Jacobian of the n-ic and two N-ics of the pencil meets the n-ic at the points of contact required, six times at each node and cusp of the n-ic. and twice at the other r base-points of the pencil. The number of points of contact is therefore

$$(2N+n-3)\,n-6\,(\delta+\kappa)-2\,r, \quad \text{while} \quad r+\delta+\kappa = \tfrac{1}{2}N(N+3)-1.]$$

Ex. 10. Show that the result of Ex. 9 is true if the n-ic has multiple points, the N-ics having a $(k-1)$-ple point at each k-ple point of the n-ic; where $$D = \tfrac{1}{2}(n-1)(n-2)-\Sigma\,\tfrac{1}{2}k\,(k-1).$$

[Use Ex. 6.]

Ex. 11. A pencil of N-ics has as base-points points of an n-ic. Any curve of the pencil meets the n-ic at p points other than the base-points, and q curves of the pencil touch the n-ic at a point which is not a base-point. Show that $q = 2p+2(D-1)-\kappa'$, where κ' is the number of cusps of the n-ic which are not base points of the pencil.

[By Ex. 4 to 7 the q points of contact are the intersections of the Jacobian of the n-ic and two N-ics with the n-ic, less each node base-point six times, each cusp base-point six times, each other base-point twice, each other node twice, and each other cusp (κ' in number) thrice.]

Ex 12. Show that the result of Ex. 11 still holds, if the n-ic has ordinary multiple points with distinct tangents at some of which the N-ics have ordinary multiple points of assigned order.

Ex. 13. The number of curves of a singly infinite family with characteristic (p, l) which touch a given curve $F = 0$ of degree n and class m is in general $nl + mp$.

[This example is a generalization of Ex. 8. For the definition see Ch. IV, § 8.

If $f(x, y, z, a) = 0$ is the family, and $\phi(\lambda, \mu, \nu, a) = 0$ its tangential equation. the points of contact of a curve of the family with the line $\lambda x + \mu y + \nu z = 0$ are the intersections of this line with the curve obtained by eliminating a between $f = 0$ and $\phi = 0$, say $\psi(\lambda, \mu, \nu, x, y, z) = 0$; where ψ is homogeneous of degree p in λ, μ, ν and degree l in x, y, z. Then the points of contact of a curve of the family with $F = 0$ are the intersections of $F = 0$ with the $(l+pn-p)$-ic

$$\psi\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}, x, y, z\right) = 0.$$

To discuss the intersections of $F = 0$ and $\psi = 0$ at a node, take the node as $(0, 0, 1)$.]

Ex. 14. If $24n^2+25$ is a square number, the locus of the nodes of those $\tfrac{1}{2}\{\sqrt{(24n^2+25)}-3\}$-ics which go through the $3n^2$ intersections of three n-ics taken in pairs is the Jacobian of the n-ics.

[The family whose nodal locus we require is the second family of Ex. 3. The two simplest cases are $n = 2$ and $n = 5$. The reader may find other cases.]

CHAPTER VIII

PLÜCKER'S NUMBERS

§ 1. Plücker's Numbers.

We shall use the following notation for any curve throughout the book unless otherwise stated:

$n \equiv$ degree, $m \equiv$ class, $\delta \equiv$ number of nodes, $\kappa \equiv$ number of cusps, $\tau \equiv$ number of bitangents, $\iota \equiv$ number of inflexions.

The six quantities n, m, δ, κ, τ, ι are called the *Plücker's numbers* of the curve. They are not independent, but are connected by various relations, of which the most useful are perhaps

(i) $m = n(n-1)-2\delta-3\kappa$,

(ii) $n = m(m-1)-2\tau-3\iota$,

(iii) $\iota = 3n(n-2)-6\delta-8\kappa$,

(iv) $\kappa = 3m(m-2)-6\tau-8\iota$,

(v) $\frac{1}{2}n(n+3)-\delta-2\kappa = \frac{1}{2}m(m+3)-\tau-2\iota$,

(vi) $\frac{1}{2}(n-1)(n-2)-\delta-\kappa = \frac{1}{2}(m-1)(m-2)-\tau-\iota$,

(vii) $\iota-\kappa = 3(m-n)$,

(viii) $2(\tau-\delta) = (m-n)(m+n-9)$,

(ix) $n^2-2\delta-3\kappa = m^2-2\tau-3\iota$.

Of these *Plücker's equations* only three are independent. If we take three, for instance, (i), (ii), (iii), or (i), (ii), (v), the other relations may be deduced.

Thus, assuming (i), (ii), (v), we get (ix) by subtracting (ii) from (i). Then, subtracting (v) from (ix), we get (vi); and so on.

The Plücker's numbers of the reciprocal curve are by Ch. IV, § 7

$$m,\ n,\ \tau,\ \iota,\ \delta,\ \kappa.$$

Hence, if we prove (i), (ii) follows at once on consideration of the reciprocal of the given curve.

Similarly, (iv) follows from (iii). The equations (v) to (ix) do not give any fresh equations by consideration of the reciprocal curve.

It remains now to establish three of Plücker's equations.

We proved (i) in Ch. VII, § 5, by considering the intersections of the first polar curve of any point O with the given curve. These intersections lie one at each point of contact of a tangent from O, two at each node, and three at each cusp of the given curve.

Then (ii) follows from the reciprocal curve.

We proved (iii) in Ch. VII, § 7, by considering the intersections of the Hessian with the given curve. These intersections lie one at each inflexion, six at each node, and eight at each cusp of the given curve.

We may deduce (v) from (i), (ii), (iii), which we have proved. But we have given an independent proof of (v) in Ch. IV, § 8; and we may deduce (iii) from (i), (ii), (v), if preferred.

It is readily seen that a curve has the same Plücker's numbers as any projection of the curve; but that curves with the same Plücker's numbers are not necessarily projections of each other. All curves with the same Plücker's numbers may be said to belong to the same *type*.

§ 2. Deficiency.

We shall denote

$$\tfrac{1}{2}(n-1)(n-2)-\delta-\kappa$$

by the symbol D, which is called the *deficiency* (or *genus*) of the curve.

We shall prove in Ch. X, § 3, that D cannot be negative; and that, if D is zero, the coordinates of any point on the curve can be expressed rationally in terms of a parameter; while conversely, if the coordinates can be so expressed, $D=0$.

For instance, a cubic cannot have two double points unless it degenerates. For the line joining two double points meets the curve twice at each, whereas a straight line meets a cubic in only three points.

Also if (a, b) is a double point of a cubic, the line

$$y-b=t(x-a)$$

meets the cubic again in a point whose coordinates are rational functions of t.

Again, a quartic cannot have four double points. For the conic through four double points and any other point of a curve meets it in nine points at least.

Also the pencil of conics through three double points of a quartic and any other given point of the curve meets the quartic again in an eighth point, whose coordinates are rational functions of the parameter of the pencil.

Equation (vi) of § 1 states that the deficiency of a curve and its reciprocal are the same. For a more general theorem of which this is a special case see Ch. XXI, § 3.

For a more general definition of 'deficiency' when the curve has multiple points other than ordinary nodes and cusps see Ch. IX, § 7.

Ex. 1. Prove the rest of equations (i) to (ix) in § 1.

Ex. 2. Prove that

$$\tau = \delta + \tfrac{1}{2}(n^2-2n-2\delta-3\kappa)(n^2-9-2\delta-3\kappa).$$

Ex. 3. Enumerate the types of cubic and quartic.

[See Ch. XIII, § 1, and Ch. XVII, § 1.]

Ex. 4. (i) If $n = m$, or if $\kappa = \iota$, we have $n = m$, $\kappa = \iota$, and $\delta = \tau$. The curve is then of the same type as its reciprocal. If n is given, the number of such types is the integral part of $\frac{1}{8}(n^2-5n+12)$.

(ii) Find the type of curve for which $n = m = \delta = \kappa = \tau = \iota$.

[(i) Use § 1 (vii), (viii).

$$\delta = \tfrac{1}{2}(n-2)(n-3)-3D, \qquad \kappa = n-2+2D.$$

For example, the cubic with a cusp, the quartic with a node and two cusps, the quintic with three nodes and three cusps, the quintic with five cusps, &c. (ii) $n = 7$.]

Ex. 5. If $\delta = \tau$, either $n = m$, $\kappa = \iota$; or else the curve is a non-singular cubic, a quartic with two nodes and a cusp, a quintic with two nodes and four cusps, or a sextic with nine cusps.

[Use § 1 (vii), (viii). The exceptions are given by $\delta = \tau$, $n+m = 9$.]

Ex. 6. If $n > m$, then $\kappa > \iota$ and $\delta >, =, < \tau$ as $n+m >, =, < 9$.

Ex. 7. If $\iota = n+m$, $\kappa = 4(D-1)$.

Ex. 8. If $\delta = 0$, $\kappa = 1$, $\tau = \frac{1}{2}(n+1)(n-3)(n^2-12)$.

Ex. 9. $m \geqslant \frac{1}{2}\{1+(4n+1)^{\frac{1}{2}}\}$.

[For $m(m-1) \geqslant n$.]

Ex. 10. (i) $m = 2(n-1)+2D-\kappa$, $\qquad \iota = 3(n-2)+6D-2\kappa$,

$$\delta = \tfrac{1}{2}(n-1)(n-2)-D-\kappa.$$

(ii) If $m = \delta$, $6D = (n-1)(n-6)$. If $2m = \iota$, $2D = n+2$.

If $2\delta = \iota$, $8D = (n-2)(n-4)$.

Ex. 11. The number of cusps cannot exceed the smaller of

$$\tfrac{3}{2}(n-2+2D) \quad \text{and} \quad \tfrac{1}{2}\{4n-3-(4n+1)^{\frac{1}{2}}+4D\}.$$

[Use Ex. 9, 10, and $\iota \geqslant 0$.]

Ex. 12. If $D = 0$ and $n > 4$, not all the double points are cusps.

Ex. 13. If $D = 0$, $m \leqslant 2(n-1)$; while $m \geqslant \frac{1}{2}(n+2)$ if n is even, and $m \geqslant \frac{1}{2}(n+3)$ if n is odd.

[If $D = 0$, $m = 2(n-1)-\kappa = \frac{1}{2}(n+2+\iota)$.]

Ex. 14. Through a fixed point O any line is drawn meeting a fixed curve in P and a fixed line in Q. If the range $(OPQR)$ has a fixed cross-ratio, the locus of R is a curve of the same type as the fixed curve.

[Project the fixed line to infinity; or note that the two curves are in plane perspective.]

Ex. 15. The result of Ex. 14 holds if we replace the fixed line by a fixed conic through O.

[Project the conic into a circle, and invert with respect to O.]

§ 3. Multiple Points with Distinct Tangents.

So far we have supposed the curve to possess only ordinary cusps and nodes.

Let us now suppose that the curve has one or more multiple points with distinct tangents. We shall show that:

Plücker's equations still hold, if we reckon each ordinary multiple point with distinct tangents as equivalent to $\frac{1}{2}k(k-1)$ *nodes and each ordinary* k'*-ple tangent with* k' *distinct points of contact as equivalent to* $\frac{1}{2}k'(k'-1)$ *bitangents in the evaluation of* δ, τ, *and* D.

One of the statements in this theorem is the reciprocal of the other. It will be sufficient to prove one of them.

We require to show that any first polar curve and the Hessian meet the given curve $k(k-1)$ times and $3k(k-1)$ times respectively at a k-ple point of the curve. For, if that is so, equations (i), (ii), (iii) of § 1 hold good when each k-ple point is considered equivalent to $\frac{1}{2}k(k-1)$ nodes; and from these three equations the other equations of § 1 can be deduced.

The result is plausible; for a curve which has k real branches all nearly passing through O has evidently $\frac{1}{2}k(k-1)$ crunodes in the neighbourhood of O; and these coalesce at O, if the branches are all made to approach O.

Suppose the curve has a k-ple point O at $(0, 0, 1)$. Its equation is

$$f \equiv z^{n-k}u_k + z^{n-k-1}u_{k-1} + \dots + u_n = 0,$$

where u_r is homogeneous of degree r in x and y.

The first polar curve of $(0, 1, 0)$ is $\frac{\partial f}{\partial y} = 0$, which has evidently a $(k-1)$-ple point at O, the tangents to the curve and the first polar at O being all distinct.

Hence by Ch. I, § 7, or Ch. VI, § 2, the curve and first polar meet in $k(k-1)$ points coinciding with O, as required.

To find the number of intersections of curve and Hessian at O we adopt an indirect method. (See also Ex. 2, below.) The number is evidently dependent solely on the nature of the curve in the neighbourhood of O; and there is therefore no loss of generality in supposing that the curve has besides O only ordinary point or line singularities, namely δ_2 nodes, κ cusps, τ bitangents, and ι inflexional tangents.

Then from the reciprocal curve as in § 1

$$n = m(m-1)-2\tau-3\iota, \quad \kappa = 3m(m-2)-6\tau-8\iota;$$

and we have just proved that

$$m = n(n-1)-2\delta_2-3\kappa-k(k-1).$$

These give

$$\iota = 3n(n-2)-6\delta_2-8\kappa-3k(k-1),$$

and therefore the curve and Hessian meet $3k(k-1)$ times at O.

The first polar of O has evidently a k-ple point at O with the same tangents as the given curve. Hence the curve and first polar meet $k(k+1)$ times at O (Ch. I, § 7; Ch. VI, § 2, (vi)); so that $2k$ of the tangents from a k-ple point must be considered as coinciding with tangents at that point.

Ex. 1. The deficiency of a curve with no singularity other than ordinary multiple points with distinct tangents is $\frac{1}{2}(m-2n+2)$.

[$\frac{1}{2}(m-2n+2) = \frac{1}{2}(n-1)(n-2)-\delta$, since $\kappa = 0$.]

Ex. 2. Establish directly the fact that the Hessian meets the curve $3k(k-1)$ times at a k-ple point.

[If $u = 0$ are the tangents to the curve at the multiple point (0, 0, 1), the tangents to the Hessian at the point are

$$u = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2;$$

by Ch. VII, § 7, Ex. 14 (iii). Hence curve and Hessian meet

$$k(k+1)+2k(k-2) = 3k(k-1)$$

times at (0, 0, 1). A similar method will apply to § 4.]

§ 4. Multiple Points with Superlinear Branches.

We show now more generally that:

Plücker's equations hold if (i) *each multiple point of order k having l ordinary superlinear branches with distinct tangents is counted as equivalent to $\frac{1}{2}k(k-3)+l$ nodes and $k-l$ cusps; and* (ii) *each multiple tangent meeting the curve in $k'+l'$ points at its l' points of contact** *is counted as equivalent to $\frac{1}{2}k'(k'-3)+l'$ bitangents and $k'-l'$ inflexions. The deficiency is taken as $\frac{1}{2}(n-1)(n-2)-\frac{1}{2}\Sigma k(k-1)$.*

---

* It is supposed that the tangent touches only one branch of the curve at each of these l' points.

By an 'ordinary' superlinear branch of order q we mean one whose Cartesian equation near the origin (taken at the multiple point) is of the form

$$y = x\left(a + bx^{\frac{1}{q}} + cx^{\frac{2}{q}} + dx^{\frac{3}{q}} + \dots\right),$$

where there is no special relation between a, b, c, d, ... , except that a may be zero, if the tangent to the branch is taken as $y = 0$ (Ch. VI, § 1).

The proof of the theorem is similar to that of § 3. We only note the modifications which are necessary.

As before, (ii) is the reciprocal of (i).

Since a factor repeated r times in u_k is repeated $r-1$ times in $\frac{\partial u_k}{\partial y}$, every tangent to a superlinear branch of the curve is a tangent to a superlinear branch of any first polar of order lower by unity. Hence (Ch. VI, § 2, Ex. 8) the curve and the first polar meet in

$$k^2 - l = 2\left\{\tfrac{1}{2}k(k-3) + l\right\} + 3(k-l)$$

points at O.

Again, to prove that the number of intersections of curve and Hessian at O is

$$6\left\{\tfrac{1}{2}k(k-3) + l\right\} + 8(k-l),$$

we may suppose that the curve has besides O no singularities except δ_2 nodes, κ_2 cusps, τ_2 bitangents, and ι_2 inflexions. As before,

$$n = m(m-1) - 2\tau_2 - 3\iota_2$$

$$\text{and} \quad m = n(n-1) - 2\delta_2 - 3\kappa_2 - (k^2 - l):$$

while

$$\kappa_2 = 3m(m-2) - 6\tau_2 - 8\iota_2 - (k-l),$$

since by Ch. VII, § 7, Ex. 4 the Hessian of the reciprocal curve meets it $k-l$ times at its points of contact with the multiple tangent which is the reciprocal of O. We deduce

$$\iota_2 = 3n(n-2) - 6\delta_2 - 8\kappa_2 - 6\left\{\tfrac{1}{2}k(k-3) + l\right\} - 8(k-l);$$

which establishes the result.

Ex. Find Plücker's numbers for a curve of degree n with a single superlinear branch of order (i) $n-1$, (ii) $n-2$.

[(i) $m = n$, $\delta = \tau = \frac{1}{2}(n-2)(n-3)$, $\kappa = \iota = n-2$, $D = 0$.

(ii) $m = 3(n-1)$, $\delta = \frac{1}{2}(n-3)(n-4)$, $\kappa = n-3$, $\tau = \frac{1}{2}(n-3)(9n-16)$ $\iota = 7n-12$, $D = n-2$.]

§ 5. Higher Singularities.

The general problem, special cases of which have been discussed in §§ 3 and 4, is as follows:

Given a curve with any multiple point whatever O, how many nodes, cusps, bitangents, and inflexions * must be considered as coinciding with O, in order that Plücker's equations may formally hold?

Suppose the numbers required are δ_1, κ_1, τ_1, ι_1. Let any first polar curve and the Hessian meet the given curve in α and β points coinciding with O respectively. Then we must have

$$2\delta_1+3\kappa_1=\alpha,\quad 6\delta_1+8\kappa_1+\iota_1=\beta \quad . \quad . \quad . \quad \text{(i)}.$$

Similarly, if for the reciprocal curve the line corresponding to O passes through ρ intersections of curve and first polar and σ intersections of curve and Hessian,

$$2\tau_1+3\iota_1=\rho,\quad 6\tau_1+8\iota_1+\kappa_1=\sigma \quad . \quad . \quad . \quad \text{(ii)}.$$

First of all it is to be noted that (i) and (ii) give identically

$$3(\alpha+\rho)=\beta+\sigma.$$

Hence the quantities δ_1, κ_1, τ_1, ι_1 could not be found if this relation were not satisfied. To show that it is satisfied, take δ_2, κ_2, τ_2, ι_2 as the number of nodes, cusps, bitangents, and inflexions not coinciding with O.†

Since the curve and the first polar of P meet altogether in $n(n-1)$ points, namely at the points of contact of the m tangents from P, twice at each of the δ_2 nodes, thrice at each of the κ_2 cusps, and α times at O,

$$\alpha=n(n-1)-m-2\delta_2-3\kappa_2.$$

Similarly from the Hessian,

$$\beta=3n(n-2)-6\delta_2-8\kappa_2-\iota_2.$$

Likewise from the reciprocal curve

$$\rho=m(m-1)-n-2\tau_2-3\iota_2,$$
$$\sigma=3m(m-2)-6\tau_2-8\iota_2-\kappa_2.$$

The last four equations give $3(\alpha+\rho)=\beta+\sigma$, as required.

The equations (i) and (ii) are then consistent but not independent. Our problem is soluble but not definite; there are an infinity of solutions. We could make the solution definite if we had a definition of the deficiency D, which was

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* i.e. How many bitangents have both points of contact at O, and how many inflexional tangents have their points of contact at O?

† We may consider these as ordinary nodes, &c., without loss of generality; for α, β, ρ, σ only depend on the nature of the curve in the immediate neighbourhood of O.

applicable to singularities other than ordinary nodes and cusps, for then we could add to equations (i) and (ii)

$$\tfrac{1}{2}(n-1)(n-2)-(\delta_1+\delta_2+\kappa_1+\kappa_2) = D.$$

Such a definition will be given in Ch. IX, § 7.

We have not proved that $\delta_1, \kappa_1, \tau_1, \iota_1$ thus obtained are positive integers or zero.

In § 4 we proved that for the case there considered

$$\alpha = k^2-l, \quad \rho = 0, \quad \sigma = k-l,$$

and verified that (i) and (ii) were satisfied by

$$\delta_1 = \tfrac{1}{2}k(k-3)+l, \quad \kappa_1 = k-l, \quad \tau_1 = \iota_1 = 0.$$

In this case the solution was made definite by our knowledge of the fact that the reciprocal curve had no multiple point on the line corresponding to O.

Ex. 1. Discuss the singularity of $y^p z^q = x^{p+q}$ at (0, 0, 1).

[Using Ch. IV, § 7, Ex. 4 and Ch. VII, § 7, Ex. 2 (i) we find

$$\alpha = (p-1)(p+q), \qquad \beta = 3p(p+q)-4p-2q,$$
$$\rho = (q-1)(p+q), \qquad \sigma = 3q(p+q)-2p-4q.$$

These satisfy $3(\alpha+\rho) = \beta+\sigma$, and § 8 (i) and (ii) give

$$\delta_1 = \tfrac{1}{2}p(p+q)-2p-\tfrac{1}{2}q+\tfrac{3}{2}F, \qquad \kappa_1 = p-F,$$
$$\tau_1 = \tfrac{1}{2}q(p+q)-\tfrac{1}{2}p-2q+\tfrac{3}{2}F, \qquad \iota_1 = q-F;$$

where F is any symmetric function of p and q.

We may determine F by noticing that, since the curve is unicursal (Ch. X, § 4, Ex. 12), the reciprocal singularities at (0, 0, 1) and (0, 1, 0) must be together equivalent to $\tfrac{1}{2}(p+q-1)(p+q-2)$ nodes and cusps; i.e. $$\delta_1+\kappa_1+\tau_1+\iota_1 = \tfrac{1}{2}(p+q-1)(p+q-2).$$

Hence $F = 1$ and

$$\delta_1 = \tfrac{1}{2}(p-1)(p+q-3), \quad \kappa_1 = p-1, \quad \tau_1 = \tfrac{1}{2}(q-1)(p+q-3), \quad \iota_1 = q-1.]$$

Ex. 2. Discuss the singularity of $(yz+x^2)^2 = y^3x$ at (0, 0, 1).

[By Ch. XVII, § 8 (V) the curve has a single point singularity at (0, 0, 1) and three other inflexions. The reciprocal curve is a 5-ic with the reciprocal singularity, no other inflexions, and three other cusps. Hence $\alpha = \rho = 7$, $\beta = \sigma = 21$. These give

$$\delta_1 = \tau_1 = 2+3k, \qquad \kappa_1 = \iota_1 = 1-2k.$$

We may prove $k = 0$, by noticing that, since the curve is unicursal, $\delta_1+\kappa_1 = 3$.]

CHAPTER IX

QUADRATIC TRANSFORMATION

§ 1. Definition of Quadratic Transformation.

SUPPOSE we have a fixed conic Σ with fixed tangents CA, CB (Fig. 1). Let any variable transversal through C cut Σ in Q, R and let (PP', QR) be an harmonic range on the line CQR. In Fig. 2, if P lies in any portion of the plane, P' lies in the other portion labelled with the same letter, as is easily verified.

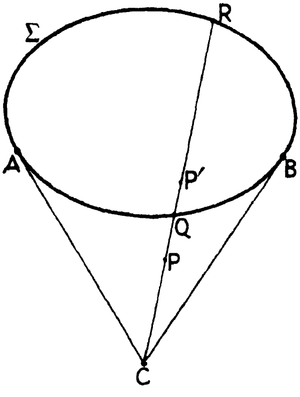

Fig. 1.

The dotted conic in Fig. 2 is the locus of the middle point of the chord QR.

Suppose that P traces out a locus c, P' will trace a locus c'. The loci c, c' are said to be derived from one another by *quadratic transformation*; Σ being the 'base-conic' and C the pole of the transformation.

Choosing ABC as triangle of reference, we may take homogeneous coordinates so that Σ is $z^2 = xy$.

If P' is the point (X, Y, Z), P is $\left(\frac{1}{Y}, \frac{1}{X}, \frac{1}{Z}\right)$; for these points are collinear with C and conjugate with respect to $z^2 = xy$.

Hence, if $f(x, y, z) = 0$ is the locus c of P, then

$$f\left(\frac{1}{y}, \frac{1}{x}, \frac{1}{z}\right) = 0$$

is the locus c' of P'.*

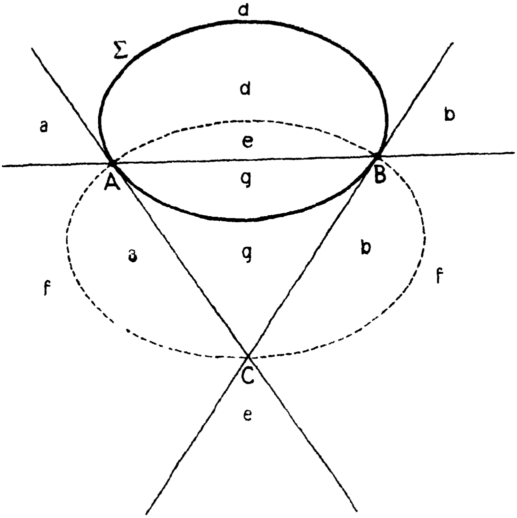

Fig. 2.

From this result, or from the original definition, we at once derive the following:

If P is on AB, P' is at C.
If P is on BC, P' is at B.
If P is on AC, P' is at A.
If c is a line through C, c' is the same line.
If c is a line through A, c' is a line through B.
If c is a line through B, c' is a line through A.
If c is any other line, c' is a conic through A, B, C.

For instance, if c is the line $\lambda x + \mu y + \nu z = 0$, c' is

$$\lambda/y + \mu/x + \nu/z = 0;$$

which is a conic through A, B, C.

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* It is more usual for reasons of symmetry to take the transformation in the form $X : Y : Z = 1/x : 1/y : 1/z$ so that the curve $f(x, y, z) = 0$ is transformed into $f(1/x, 1/y, 1/z) = 0$. This is equivalent to quadratic transformation followed by interchange of the vertices A and B of the triangle of reference.

Again:

If c meets AB at P, c' touches CP at C.*

To the tangent at P to c corresponds the conic osculating c' at C and passing through A and B.

For if Q is close to P on c, Q' is on c' close to C, and the limiting position of CQQ' (which is CP) is the tangent to c' at C.

Also, to the line joining P to any point R of c corresponds a conic through A, B, R' touching c' at C. If R approaches P, the conic becomes the osculating conic to c' at C.

Again:

If c meets CA at P, c' touches at A the line through A corresponding to BP, i.e. the line joining A to the second intersection of BP with Σ. Similarly if c meets CB at P.

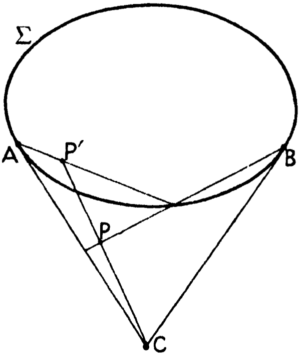

Fig. 3.

For, if P is close to CA, P' is close to A (Fig. 3), and AP', BP meet on Σ.

Hence to every intersection of c with AB corresponds a branch of c' through C, to every intersection of c with CA (or CB) corresponds a branch of c' through A (or B); and conversely.

If c touches AB, two tangents to c' at C coincide; and similarly if c touches CA, two tangents to c' at A coincide.

Suppose c is an n-ic with a k-ple point at C, a p-ple point at A, and a q-ple point at B.

Then c meets AB, CA, CB respectively at $n-p-q$, $n-p-k$, $n-q-k$ other points. Hence c' has $n-p-q$, $n-p-k$,

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* Provided P is not at A, B, or C; and so throughout. It follows that, if c' has k linear branches touching CP at C, P is an ordinary k-ple point of c; and vice versa.

$n-q-k$ branches respectively through C, A, B. Also it meets AB at the k intersections of AB with the tangents to c at C; and similarly it meets CA and CB at p and q points respectively other than A, B, C.

Since c' meets AB at $\{(n-p-k)+(n-q-k)+k\}$ points, it is a $(2n-p-q-k)$-ic. Hence:

If c is an n-ic with a k-ple point at C, a p-ple point at A, and a q-ple point at B, then c' is a $(2n-p-q-k)$-ic with an $(n-p-q)$-ple point at C, an $(n-p-k)$-ple point at A, and an $(n-q-k)$-ple point at B.

For instance, taking $n = 2$, $p = q = 1$, $k = 0$, we see that the transform of a conic through A and B is a conic through A and B.

Ex. 1. Properties of any quintic with three or more double points can be deduced from those of a quartic with the same deficiency.

[Take three double points as A, B, C of § 1. The properties so derived are not usually of much elegance or importance. Our knowledge of the theory of quintic curves (and still more of sextic) is very limited.]

Ex. 2. If two curves (or branches of the same curve) have r-point contact at a point on CA, the transformed curves (or branches) have $(r+1)$-point contact at A.

§ 2. Inversion.

If in § 1 we project A, B into the circular points, Σ becomes a circle with centre C, and P, P' are inverse points with respect to this circle. The loci c, c' of P, P' are now inverses of each other with respect to the circle. Hence inversion is a particular case of quadratic transformation, and quadratic transformation is the generalization by projection of inversion.

From the results of the last section we may derive many well-known theorems connected with inversion. For instance, 'the inverse of a circle is a circle', 'the inverse of a curve with respect to a focus is a curve with a cusp at each circular point', and so on.

One of the main uses of quadratic transformation is to deduce properties of a curve c from those of its transform c'.

This is exactly equivalent to the process of projecting two points into the circular points and inverting with respect to some other point.

The main advantage of quadratic transformation over projection followed by inversion is that, if A, B in Fig. 1 are to be projected into the circular points, it is convenient that they should bear similar relations to the curve c (e. g. be both nodes, or both cusps, &c.); and this restriction is unnecessary if we employ the generalized form of inversion, namely, quadratic transformation.

Quadratic transformation (or the equivalent process of inversion) will also enable us to simplify the solution of problems which have been discussed in Ch. VI and elsewhere, such as the determination of the number of intersections of two curves at a given point, the expansion of y in terms of x near a given point, the number of tangents from a point whose points of contact coincide with the point, &c.

§ 3. The Number of Intersections of two Curves at a given Point.

If the point P in Fig. 1 does not lie on a side of the triangle ABC, the quadratic transformation evidently transforms two curves meeting at P into two curves meeting at P'. Suppose then we have two curves of degrees n and N, and we want to know how many intersections of these curves must be considered as lying at a certain point, which is a k-ple and a K-ple point on the two curves respectively, if we are to observe the convention that an n-ic and an N-ic meet at nN points.

Take the point as C, and take the sides of the triangle ABC (Fig. 1) so as to have no other special relation to either curve. Suppose that the curves meet in r points other than C, and therefore $nN-r$ times at C. The transformed curves are of degrees $2n-k$ and $2N-K$, have multiple points of orders $n-k$ and $N-K$ at each of A and B, and have multiple points of orders n and N at C (§ 1).

At A, B, and C no two branches of the transformed curves touch one another; since we took the points A and B in a general position. The transformed curves therefore meet $(n-k)(N-K)$ times at each of A and B, and nN times at C (Ch. I, § 7; Ch. VI, § 2). They meet also at the transforms of the r intersections other than C of the original curves. Hence they must meet at s points on AB other than A and B, where

$$s = (2n-k)(2N-K)-2(n-k)(N-K)-nN-r$$
$$= nN-kK-r.$$

Therefore $nN-r = s+kK$, so that:

If two curves have a k-ple and a K-ple point at C respectively, and the transformed curves meet at s points on the line AB other than A and B, the two given curves meet s+kK times at C.

We suppose, as stated before, that the lines CA and CB have no special relation to the curve, though this restriction may at times be removed if due precautions are taken.

If the transformed curves intersect at a point H on AB, and it is not immediately evident how many times they meet at H, we may apply the above theorem to the transformed curves, taking C at H.

As a simple example of the above theorem, take the case of a curve passing through the cusp C of another, the two curves having a common tangent at C, which meets AB at H. The transformed curves pass through H, one touching AB and the other not, so that they have a simple intersection at H. Hence the number of intersections of the original curves at C is $1+2.1 = 3$.

As another example, take the case of curves with k and K linear branches at C respectively, all the branches having CH as a common tangent. The transformed curves have k-ple and K-ple points at H, and therefore meet in kK points coinciding with H. The original curves therefore meet at $2kK$ points coinciding with C.

§ 4. Class of a Curve.

Suppose we wish to find the number of intersections α of a curve with any first polar curve at a multiple point C of order k on an n-ic. We may find the equation of the polar curve and then use § 3. An alternative method is the following.

Suppose A and B have no special relation to the curve. The first polar of P meets the n-ic α times at C, and at the m points of contact of the tangents from P. We shall assume the n-ic has no multiple point other than C.* Then

$$m = n(n-1)-\alpha.$$

To each of the m tangents from A to the n-ic corresponds a tangent from B to the transformed curve whose point of contact does not lie on AB. Now the transformed curve is of degree $2n-k$. It has ordinary $(n-k)$-ple points at A and B, and an n-ple point at C. It meets the first polar of B $(n-k)(n-k-1)$ times at A, $(n-k)(n-k+1)$ times at B, $n(n-1)$ times at C (Ch. VIII, § 3), at the m points of contact of tangents from B not lying on AB, and (say) α' times at points on AB other than A and B.

---

* α depends only on the shape of the n-ic in the neighbourhood of C, so this assumption involves no loss of generality.

Therefore

$$(2n-k)(2n-k-1) = (n-k)(n-k-1)+(n-k)(n-k+1) \\ +n(n-1)+m+\alpha',$$

or $$(n-k)(n+k-1)-\alpha' = m = n(n-1)-\alpha.$$

This gives

$$\alpha = \alpha'+k(k-1).$$

Now the number of intersections at H of a curve with the first polar of O is dependent on the shape of the curve in the neighbourhood of H in general and not on the position of O (unless O is near H or on the tangent at H).

Hence:

The number of intersections of a curve with any first polar curve at a k-ple point C is equal to the number of intersections $k(k-1)$ *at an ordinary k-ple point together with the number of those intersections of the transformed curve with any first polar which lie on AB but not at A or B.*

For example, if the curve has k linear branches with a common tangent at C meeting AB at H, the transformed curve has a k-ple point at H. The effect of this on the class of the transformed curve is to lower it by $k(k-1)$; and therefore the effect of the singularity at C is to lower the class of the original curve by $2k(k-1)$.

§ 5. Tangents from a Singular Point to a Curve.

Suppose that in the curve of § 4 λ of the tangents from C have their point of contact at C, and that μ of the tangents from C to the transformed curve have their points of contact on AB. The other tangents from C to the transformed curve are the n tangents at C each counted twice, and the $m-\lambda$ tangents from C to the given n-ic which do not touch at C. Hence the class of the transformed curve is

$$m-\lambda+\mu+2n.$$

But we see, as in § 4, by consideration of the tangents from B that this class is

$$m+2(n-k)+\epsilon,$$

where ϵ is the number of tangents from B to the transformed curve whose points of contact are on AB but not at B (or A).

Therefore $\lambda = 2k+\mu-\epsilon$, or:

The number of tangents from the k-ple point C to a curve whose points of contact coincide with C is $2k$, *plus the number of tangents from C to the transformed curve whose points of* *contact are on AB, minus the number of tangents from B to the transformed curve whose points of contact are on AB but not at B.*

For example, if the curve has k linear branches at C with a common tangent meeting AB at H, the transformed curve has a k-ple point at H at which in general neither AB nor CH is a tangent. Hence the points of contact of $2k$ tangents from C coincide with C.

Ex. If a tangent at a k-ple point C of an n-ic meets the curve in $k+r$ points coinciding with C and meets AB at H, the transformed curve meets CH in r points coinciding with H.

[The transformed curve meets CH n times at C and $n-k-r$ times not at C or H. But it is of degree $2n-k$.]

§ 6. Latent Singularities.

Suppose that the transform of a curve with a k-ple point at C has a j-ple point at H on AB, not coinciding with A or B. Suppose now that the transformed curve and the lines CA, CB are kept fixed while the line AB is slightly displaced so as not to pass through H. Then the original curve will alter its shape slightly. The transformed curve now cuts AB in k distinct points not close to A or B (j of them very close to H), so that the original curve has an ordinary k-ple point at C and a j-ple point corresponding to H at the point H' very close to C. As AB is moved back into its original position the j-ple point at H' moves up to C and coalesces with the k-ple point to form a 'higher singularity' at C. The j-ple point of the transformed curve is said to be *latent* in the k-ple point C of the original curve.

We say that C has been 'analysed by quadratic transformation'.

§ 7. Deficiency.

We defined the 'deficiency' D of an n-ic in Ch. VIII, §§ 2 and 3 as $\frac{1}{2}(n-1)(n-2)-\delta-\kappa$, where δ was the number of nodes and κ the number of cusps, with the proviso that an ordinary k-ple point is to count as $\frac{1}{2}k(k-1)$ nodes.

It is desirable to give a definition which shall be valid when the curve has higher singularities. It is difficult to do this satisfactorily without a knowledge of function-theory, but an attempt is here made. (See also § 12, Ex. 10.)

First we word the definition of deficiency when no higher singularities exist a little differently. We say that the *deficiency of an n-ic is one more than the number of arbitrary coefficients in the equation of the most general adjoined* $(n-3)$*-ic.**

By an 'adjoined' N-ic we mean one which has a $(k-1)$-ple point at every k-ple point of the n-ic.

That this definition of deficiency is equivalent to the earlier one is readily proved.

To state that a curve has a $(k-1)$-ple point at a given point is equivalent to assigning $\frac{1}{2}k(k-1)$ linear relations between the coefficients of its equation (Ch. II, § 6). Hence the $(n-3)$-ic has

$$\tfrac{1}{2}(n-3)n-\Sigma\tfrac{1}{2}k(k-1)$$

arbitrary coefficients, the summation being taken over every multiple point of the n-ic. But this is the number we have defined in Ch. VIII, § 3 as $D-1$.

We shall state the fact that an N-ic has a $(k-1)$-ple point at a given k-ple point O of an n-ic, by saying that the N-ic is 'adjoined to the n-ic at O'. If the N-ic is 'adjoined' to the n-ic, it is adjoined at every multiple point.

§ 8. Deficiency unaltered by Quadratic Transformation.

We now show that the deficiency of a curve and its quadratic transform are the same, when the curve has only ordinary multiple points. Suppose as in § 1 that an n-ic has p-ple, q-ple, k-ple points at A, B, C respectively. Any adjoined $(n-3)$-ic has $(p-1)$-ple, $(q-1)$-ple, $(k-1)$-ple points at A, B, C. By § 1 the transforms of the n-ic and $(n-3)$-ic are therefore a $(2n-p-q-k)$-ic with $(n-p-k)$-ple, $(n-q-k)$-ple, $(n-p-q)$-ple points at A, B, C and a curve of degree

$$2(n-3)-(p-1)-(q-1)-(k-1) = 2n-p-q-k-3$$

with a multiple point of order

$$(n-3)-(p-1)-(q-1) = n-p-q-1$$

at C, and so for A and B.

Thus the transforms of the curve and an adjoined curve of degree lower by three are also a curve and an adjoined curve of degree lower by three; from which the equality of the deficiencies of the n-ic and its transform follows immediately.

We have assumed p, q, k all greater than zero. Now suppose $k=0$. The transform of the $(n-3)$-ic is now only of degree $2n-p-q-4$ (not $2n-p-q-3$, as required for the proof) and has a multiple point of order $n-p-2$ (not $n-p-1$) at A; and so for B. But if, as is lawful, we take as the transform of the $(n-3)$-ic the $(2n-p-q-4)$-ic together with the line AB, we obtain a curve of degree $2n-p-q-3$ adjoined to the transform of the n-ic.

---

* To complete the definition add 'and $D=0$, if $n=1$ or 2; while $D=0$ or 1 when $n=3$, according as the cubic has or has not a double point'.

In fact, since the transform of the n-ic is of degree $2n-p-q$ with $(n-p)$-ple and $(n-q)$-ple points at A and B, an adjoined $(2n-p-q-3)$-ic has $(n-p-1)$-ple and $(n-q-1)$-ple points at A and B. The line AB therefore meets the adjoined curve in points whose number is greater than the degree of the curve, showing that the curve degenerates into AB and a $(2n-p-q-4)$-ic.

Similarly, if $p = 0$. CA is part of the transformed adjoined curve; and, if $q = 0$, CB is part of the curve.

In practice, however, the case p, q, k all greater than zero is the only one we need consider. In the following we shall require to transform any curve by successive quadratic transformations into one having only ordinary multiple points with distinct tangents. To do this we can take A and B as ordinary points on the curve and C as any multiple point which is not 'ordinary'. The process of transformation is repeated till only ordinary multiple points are left.*

§ 9. Deficiency for Higher Singularities.

The definition of deficiency can be applied to curves with higher singularities when we have defined what we mean by a curve adjoined to an n-ic with such singularities.

Suppose that in § 6 a curve has a $(j-1)$-ple point at H, then its transform would be considered as adjoined to the given curve at the k-ple point C. If the j-ple point H is not an ordinary one, it could be still further analysed by taking C at H in § 6. In this way we can find what is intended by a curve adjoined to the transformed curve at H, and then the transform of this adjoined curve will be adjoined at C to the given curve.

According to this definition we shall still have deficiency unaltered by quadratic transformation.

For instance, if the given curve has k linear branches touching CH at C, the transformed curve has an ordinary k-ple point at H. A curve adjoined to the transformed curve has therefore a $(k-1)$-ple point at H. Hence a curve adjoined to the given curve has $k-1$ linear branches touching CH at C. Also the k-ple point C lowers the deficiency by $k(k-1)$, since the k-ple point at H lowers the deficiency of the transformed curve by $\frac{1}{2}k(k-1)$.

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* It is fairly evident that this is possible. We do not give here the formal proof.

§ 10. Intersections of a Curve with Adjoined Curves.

To state that an N-ic is adjoined to an n-ic at an ordinary k-ple point C is equivalent to assigning $\frac{1}{2}k(k-1)$ linear relations between the coefficients of the equation of the N-ic. The number of intersections of the n-ic and N-ic at C is $k(k-1)$, which is twice the number of linear relations just referred to.

Now suppose, as in § 6, that C has a latent j-ple point. The transform of the N-ic has an equation whose coefficients are subjected to $\frac{1}{2}j(j-1)$ linear relations, since it has a $(j-1)$-ple point at H. Hence the coefficients of the equation of the N-ic are subjected to $\frac{1}{2}j(j-1)$ linear relations in addition to the $\frac{1}{2}k(k-1)$ relations which state that C is a $(k-1)$-ple point of the N-ic.

Also, since the transforms of n-ic and N-ic meet $j(j-1)$ times at H, the n-ic and N-ic meet $k(k-1)+j(j-1)$ times at C by § 3, which is again twice the number of relations between the coefficients of the N-ic due to the fact that it is adjoined to the n-ic at C.

If H is not an ordinary j-ple point, it may be analysed in its turn till the n-ic is finally resolved into a curve with ordinary multiple points only.* Hence:

The number of intersections of an n-ic and adjoined N-ic at the multiple points of the n-ic is twice the number of relations to which the coefficients of the equation of the N-ic are subjected owing to the fact that the N-ic is adjoined to the n-ic.

The number of relations in question is

$$\tfrac{1}{2}(n-1)(n-2)-D;$$

for an adjoined $(n-3)$-ic has $D-1$ arbitrary coefficients in its equation.

We deduce that

If a pencil of N-ics is adjoined to an n-ic and the base-points of the pencil other than the multiple points of the n-ic are also on the n-ic, any N-ic of the pencil meets the n-ic in $\frac{1}{2}(n-N)(N-n+3)+D$ variable points $(n>N)$.

---

* This statement assumes that a curve and an adjoined curve are transformed into a curve and an adjoined curve; i.e. $N=n-3$. But the number of intersections and of relations between the coefficients due to the adjoining at C is evidently independent of the value of N.

Of the fixed intersections of the n-ic with a variable N-ic of the pencil $(n-1)(n-2)-2D$ lie at the multiple points and

$\frac{1}{2}N(N+3)-1-\frac{1}{2}(n-1)(n-2)+D$ [*]

at the remaining base-points of the pencil.

There remain

$$\begin{aligned} &nN-\{(n-1)(n-2)-2D\} \\ &\quad -\{\tfrac{1}{2}N(N+3)-1-\tfrac{1}{2}(n-1)(n-2)+D\} \\ &\qquad\qquad = \tfrac{1}{2}(n-N)(N-n+3)+D \end{aligned}$$

variable intersections.

If $N=n-3$, the number is D. If $N=n-1$ or $n-2$, the number is $D+1$.

Ex. 1. A pencil of N-ics has as its base-points points of an n-ic, being adjoined to it at all its singularities with the exception of κ' cusps and certain of its ordinary multiple points. If any N-ic of the pencil meets the n-ic at p points other than base-points, while q N-ics of the pencil touch the n-ic at a point other than the base-points, show that

$$q=2p+2(D-1)-\kappa'.$$

[The n-ic may be transformed by successive quadratic transformations into a curve with ordinary multiple points; and p, q, κ', D are unaltered by this process. Now use Ch. VII, § 10, Ex. 12.]

Ex. 2. A pencil of N-ics has as its base-points points of an n-ic and is adjoined to it. Show that

$$(n-N)(N-n+3)+4D-2$$

of the N-ics touch the n-ic at points other than a base-point.

§ 11. Another Transformation.

A form of transformation alternative to that in § 1 is obtained by taking as our base-conic a pair of lines through B harmonically conjugate to BA and BC. As before, we take any transversal through C cutting these lines in Q and R and take points P and P' on the transversal so that (PP', QR) is harmonic (Fig. 4).

In Fig. 5, if P lies in any portion of the plane, P' lies in the other portion labelled with the same letter, the hyperbola in the figure being the locus of the middle point of the chord QR.

Choosing homogeneous coordinates so that ABC is the triangle of reference and the two fixed lines are $x^2=z^2$, we find that, if P' is (X, Y, Z), P is $(X, Y, X^2/Z)$, since CPP' is a straight line and the pencil $B(PP', QR)$ is harmonic.

Hence if the locus of P is $f(x, y, z)=0$, the locus of P' is $f(x, y, x^2/z)=0$.

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[*] N must be such that this quantity is positive or zero. We may always have $N=n-1$ or $n-2$, when $n>2$. For N to be $n-3$, we must have $D>1$.

It is at once shown that this transformation may be obtained by taking in succession three transformations of the type of § 1, the base-conics being respectively

$$x^2 - xy + z^2 = 0, \quad x^2 - y^2 + z^2 = 0, \quad x^2 + xy - z^2 = 0$$

and the pole of the transformations being (0, 0, 1).

The properties of the transformation are somewhat similar to those of the transformation of § 1.

The reader will have no difficulty in verifying the following statements, in which c and c' are the loci of P and P' respectively:

If c is a line through C, so is c'.

If c is a line through B, so is c'.

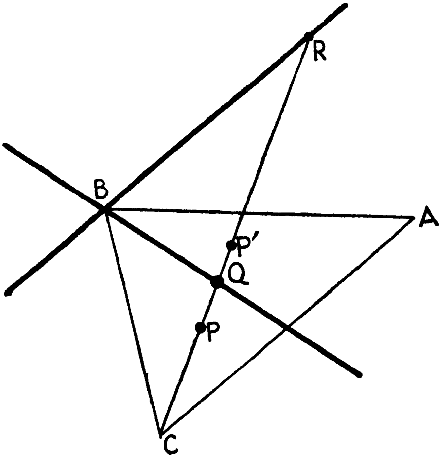

Fig. 4.

If c is any other line, c' is a conic through C touching AB at B.

If c meets AB at P, c' touches CP at C. To the tangent at P to c corresponds the conic osculating c' at C and touching AB at B.

To each intersection of c with BC corresponds a linear branch of c' touching AB at B.

To each linear branch of c through B (not touching AB) corresponds a linear branch of c' through B, the tangents to the two branches being harmonic conjugates with respect to the two given lines.

If c is an n-ic with a k-ple point at C and a q-ple point at B, c' is a $(2n-k-q)$-ic with an $(n-q)$-ple point at C and

$n-k$ linear branches through B, $n-k-q$ of which touch AB at B.

The theorems in italics in §§ 3, 4, 5 relating to the number of intersections of two curves, tangents from a singular point, and class still hold for the transformation of this section.* The proofs are very similar, and the necessary modifications may be left to the reader. He will require the application of the theorems to the case of a point at which linear branches touch. This has been given at the end of each of these sections by way of illustration.

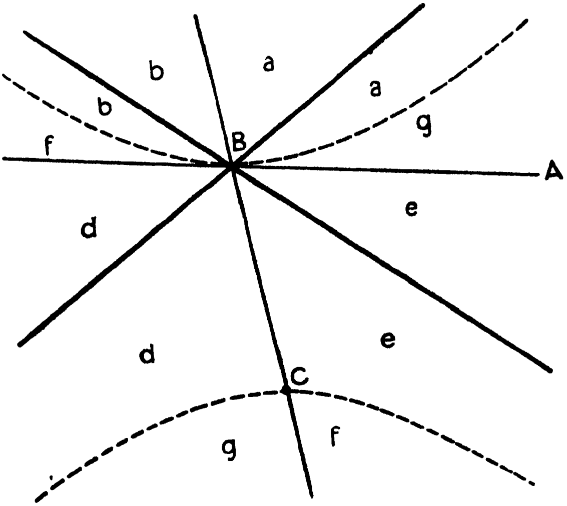

Fig. 5.

The transformation of this section does not alter the deficiency of a curve. For it is equivalent to three transformations of the type of § 1, none of which alters the deficiency (§ 8).

§ 12. Applications of this Transformation.

The main advantage of the method of § 11 is that the position of the line CA is still at our disposal. For instance, suppose the curve $f(x, y, z) = 0$ goes through C. We may take CA as a tangent at C without loss of generality, which was not the case in § 1. Much simplification in arithmetic may be thus secured.

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* In the theorem of § 3 read 'other than B' for 'other than A or B'. In the theorem of § 4 read 'but not at B' for 'but not at A or B'.

If *CA* is a tangent at *C*, the transformed curve passes through *A*. It is convenient to take *CBA* instead of *ABC* as triangle of reference for the transformed curve. This is equivalent to interchanging x and z in the equation of this transformed curve, which now becomes $f(z, y, z^2/x) = 0$, the original curve being $f(x, y, z) = 0$.

Writing it in the form $f(x/z, xy/z^2, 1) = 0$, we see that, if the Cartesian equation of the original curve is $f(x, y) = 0$, the transformed curve is $f(x, xy) = 0$.

Points of $f(x, y) = 0$ near the origin correspond to points of $f(x, xy) = 0$ near the axis of y, and points of one curve near the axis of x correspond to points of the other curve also near the axis of x.

If the nature of any singularity *H* of the transformed curve is not immediately obvious, we may take *H* as origin and repeat the transformation.

Ex. 1. If *CA* is the tangent at the cusp *C* of a curve, *AB* is an ordinary tangent to the transformed curve at A.

More generally, if *CA* is a tangent at *C* to a superlinear branch of order r, *AB* is a tangent of r-point contact to a linear branch at *A*.

[$y^r = ax^{r+1} + bx^r y + cx^{r-1}y^2 + \dots + ky^{r+1} + \dots$ becomes, on putting xy for y and dividing by x^r,

$$y^r = ax + bxy + \dots + kxy^{r+1} + \dots .]$$

Ex. 2. If *CA* is a tangent of r-point contact at *C* to a curve, *CA* is a tangent of $(r-1)$-point contact at *A* to the transformed curve.

Ex. 3. A quadruple point of a curve consists of two cusps with a common tangent. Find the effect of this point on class, deficiency, &c.

[Taking *C* as multiple point and *CA* as tangent, the transformed curve has two linear branches touching *AB* at *A* (Ex. 1). We have shown that the effect of such a 'tacnode' as *A* is to lower the class by 4, and also the two tangents from *B* to the transformed curve touch at *A*. Therefore the effect of the quadruple point on the class is to lower it by $4+2+4.3 = 18$. Again, the tacnode lowers the deficiency by 2, so that the quadruple point lowers the deficiency by $2+\frac{1}{2}4.3 = 8$. A curve adjoined to the transformed curve at *A* has a linear branch touching *AB* at *A*, so that a curve adjoined at *C* to the given curve has a cusp at *C* with *CA* as tangent. The tacnode is *latent* in the quadruple point *C*, which may be considered (Ch. VIII, § 5) as equivalent to δ nodes and κ cusps where $\delta+\kappa = 8$, $2\delta+3\kappa = 18$, or $\delta = 6$, $\kappa = 2$.

The tangents from *C* which touch at *C* are $2.4+0-2 = 6$ in number.]

Ex. 4. Discuss similarly the case of the rhamphoid cusp.

[Taking *C* as cusp and *CA* as tangent at *C* the curve is

$$0 = (y+ax^2)^2 + y^2(bx+cy) + y(dx^3+ex^2y+fxy^2+gy^3) + hx^5 + \dots ;$$

and the transformed curve is

$$0 = (y+ax)^2 + hx^3 + \dots$$

which has a cusp at *A*.

Hence C is a double point with a latent cusp. The existence of a rhamphoid cusp lowers the deficiency by 2 and the class by $2+3=5$. An adjoined curve has a linear branch touching CA at C. The rhamphoid cusp may be considered as formed by uniting an ordinary node and cusp. Since the reciprocal of a rhamphoid cusp is also a rhamphoid cusp (Ch. VI, § 5, Ex. 1), it may be considered as formed by the union of a bitangent and inflexion. The number of tangents from C whose points of contact lie at C is 4.]

Ex. 5. Discuss the nature of the origin for $y^2 = x^{2n+1}$.

[Put yx for y and use induction.
There is a latent cusp and $n-1$ latent nodes. The origin lowers the class by $2n+1$ and the deficiency by n. An adjoined curve has a linear branch with $y=0$ as a tangent of n-point contact.]

Ex. 6. Discuss the origin for the curves

$$\text{(i)}\ y^2+2x^py+x^{2p\pm1}=0.$$
$$\text{(ii)}\ y^p = x^{p\pm q1}.$$
$$\text{(iii)}\ (y-x^p)^q = x^{p\pm q1}.$$

[Put yx for y and use induction.]

Ex. 7. Show that the effect on Plücker's numbers of a double point at which two linear branches have r-point contact is the same as that of r nodes and r bitangents in general.

[Use induction.]

Ex. 8. Find any latent double points of the following curves at the origin; where $u \equiv x+y^2$.

$$\text{(i)}\ u^2 = y^2(px^2+qy^2).$$
$$\text{(ii)}\ u^2 = xy^3.$$
$$\text{(iii)}\ (u-\alpha y^2)(u-\beta y^2)(u-\gamma y^2) = k(xu-y^3)^2.$$

[(i) Two latent nodes.
(ii) A latent node and cusp. As another example find the double points latent in its Hessian at the origin.
(iii) Put in turn yx for y, $x-y$ for x, xy for x, &c. Consider separately the cases $k=1$, $k=-\alpha\beta\gamma$, and also

$$\gamma=0, \quad \alpha=\beta, \quad \beta=\gamma=0, \quad \alpha=\beta=\gamma=0.$$

To trace the curve, find its intersections with $u=ty^2$.]

Ex. 9. The order of a superlinear branch at C (§ 11) is equal to the order of the transformed branch plus the number of tangents from B to this transformed branch which coincide with BA.

[Put yx for x in the expansion (i) of Ch. VI, § 3. If $2\alpha > \beta > \alpha$, apply 'reversion of series' (Ch. VI, § 1) and the theorem at the end of Ch. VI, § 5 to the transformed expansion.]

Ex. 10. If l is the order of any superlinear branch of a curve,

$$2(D-1) = m-2n+\Sigma(l-1),$$

the summation being taken over all the superlinear branches of the curve.

[Repetitions of the transformation of § 11 will transform the curve eventually into a curve with no superlinear branch, for which the result is readily seen to be true. But Ex. 9 proves $m-2n+\Sigma(l-1)$ unaltered by such a transformation.

We may define the deficiency as

$$\tfrac{1}{2}\{m-2n+\Sigma(l-1)\}+1,$$

if we choose. This is an alternative definition to that given in § 7.]

Ex. 11. Prove Ch. VI, § 2, Ex. 7 by quadratic transformation.

[Use Ex. 2, § 3, and induction.]

Ex. 12. Two curves having at C superlinear branches of orders k and K $(K \geqslant k)$ with a common tangent meet $k(K+1)$ times at C.

[Use Ex. 1 and 11.]

Ex. 13. Two curves have a common tacnode C with a common tangent at C. How many of their intersections coincide with C? Discuss the case in which two or more of the branches at C osculate.

Ex. 14 Discuss similarly the case in which C is a rhamphoid cusp of both curves.

CHAPTER X

THE PARAMETER

§ 1. Point-coordinates in terms of a Parameter.

SUPPOSE the coordinates of any point (x, y, z) of a curve given as functions of a quantity t, which we will call the *parameter* of the point, by the equations

$$x = f(t), \quad y = \phi(t), \quad z = \psi(t).$$

The equation of the line joining the points with parameters t, t_1 is evidently

$$\begin{vmatrix} x & y & z \\ f(t) & \phi(t) & \psi(t) \\ \dfrac{f(t_1) - f(t)}{t_1 - t} & \dfrac{\phi(t_1) - \phi(t)}{t_1 - t} & \dfrac{\psi(t_1) - \psi(t)}{t_1 - t} \end{vmatrix} = 0 \quad . \quad . \quad \text{(i)}.$$

Making t_1 approach t, we have the equation

$$\begin{vmatrix} x & y & z \\ f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

for the tangent at the point with parameter t.

The condition that the points with parameters t, t_1, t_2 should be collinear is

$$\frac{1}{(t_1 - t_2)(t_2 - t)(t - t_1)} \begin{vmatrix} f(t) & \phi(t) & \psi(t) \\ f(t_1) & \phi(t_1) & \psi(t_1) \\ f(t_2) & \phi(t_2) & \psi(t_2) \end{vmatrix} = 0 \quad . \quad \text{(iii)}.$$

To find the real nodes, we suppose that the parameter of a point considered as lying on one branch of the curve at a node is t, and that the parameter of the point considered as lying on the other branch is t_1. The equation (iii) must be satisfied for all values of t_2, since the points with parameters t and t_1 coincide.

This will give us equations in t and t_1 which are equivalent to two equations to solve for t and t_1.*

---

* We assume throughout that in general to each point on the curve corresponds a single value of t. See § 4, Ex. 13.

A similar method is to equate to zero the coefficients of x, y, z in (i); for the line joining two coincident points is indeterminate.

If the values of t and t_1 thus obtained are real,* the node is a crunode; for on giving a small increment to the value of t or t_1 we obtain a real point on the curve close to the node.

If the values of t and t_1 are conjugate complex quantities, the node is a real acnode.

The cusps are given by the equations

$$f'(t)/f(t) = \phi'(t)/\phi(t) = \psi'(t)/\psi(t) \quad . \quad . \quad . \quad \text{(iv)}.$$

For if the values of the parameter at a double point are t and t_1,

$$f(t_1)/f(t) = \phi(t_1)/\phi(t) = \psi(t_1)/\psi(t) = k \text{ (say)}.$$

Therefore

$$(k-1)/(t_1-t) = \{f(t_1)-f(t)\}/(t_1-t)f(t) = f'(t+\theta[t_1-t])/f(t),$$

where $1 \geqslant \theta \geqslant 0$, by the mean-value theorem.

Now make t_1 approach t, i. e. suppose the tangents at the double point approach coincidence. Then

$$f'(t)/f(t) \text{ and similarly } \phi'(t)/\phi(t),\ \psi'(t)/\psi(t)$$

all become equal to the limiting value of $(k-1)/(t_1-t)$.

The inflexions are given by the equation

$$F(t) \equiv \begin{vmatrix} f & \phi & \psi \\ f' & \phi' & \psi' \\ f'' & \phi'' & \psi'' \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad \text{(v)};$$

where $f, f', \ldots$ denote $f(t), f'(t), \ldots$.

For suppose the tangent (ii) meets the curve again in the point with parameter T. Then

$$\begin{vmatrix} f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \\ f(T) & \phi(T) & \psi(T) \end{vmatrix} = 0.$$

We may replace the elements in the third row of the determinant by $2\{f(T)-f(t)-(T-t)\,.\,f'(t)\}/(T-t)^2$, &c., without altering its value.

Now if the point of contact approaches an inflexion, one value of T approaches t. But

$$\mathop{\mathrm{L}^{t}}_{T\to t} 2\{f(T)-f(t)-(T-t)\,.\,f'(t)\}/(T-t)^2 = f''(t);$$

which proves the result.

---

* It is supposed that f, ϕ, ψ are real, if t is real.

The cusps, for which $f'/f = \phi'/\phi = \psi'/\psi$, are also given by $F(t) = 0$. The cusps are given by the values of t satisfying both $F(t) = 0$ and

$$F'(t) \equiv \begin{vmatrix} f & \phi & \psi \\ f' & \phi' & \psi' \\ f''' & \phi''' & \psi''' \end{vmatrix} = 0;$$

while the parameters of the inflexions satisfy $F(t) = 0$, but not $F'(t) = 0$ in general.

If f, ϕ, ψ are polynomials in t, the cusps are given by the repeated roots of $F(t) = 0$, and the inflexions by the single roots.

The degree of the curve (if it is algebraic) is obtained by noting that the line

$$\lambda x + \mu y + \nu z = 0$$

meets the curve where

$$\lambda f(t) + \mu\phi(t) + \nu\psi(t) = 0.$$

If this equation gives values of t corresponding to exactly n distinct points on the curve, the curve is of degree n.

§ 2. Line-coordinates in terms of a Parameter.

If

$$\lambda x + \mu y + \nu z = 0$$

is a tangent to the curve in § 1, we may take

$$\lambda = \phi\psi' - \phi'\psi, \quad \mu = \psi f' - \psi' f, \quad \nu = f\phi' - f'\phi$$

by equation (ii) of § 1.

Hence the line-coordinates λ, μ, ν of any tangent to the curve are expressed in terms of the parameter t.

Just as we obtained the parameters of the nodes and cusps and the degree of the curve, so by using the line-coordinates we may obtain the parameters of the bitangents and inflexional tangents and the class of the curve.

For instance, the inflexional tangents will be given by $\lambda'/\lambda = \mu'/\mu = \nu'/\nu$; which correspond to equation (iv) of § 1.

An alternative method of finding the nodes and bitangents is as follows. Suppose that the tangent (ii) of § 1 meets the curve again at a point whose parameter is T. Then

$$\frac{1}{(T-t)^2} \begin{vmatrix} f(T) & \phi(T) & \psi(T) \\ f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \end{vmatrix} = 0 \quad . \quad . \quad . \quad \text{(i)}.$$

Write down the condition that this equation in T has equal roots. We obtain an equation in t whose solutions are the parameters of the points of contact of the bitangents and the points of contact of the tangents from the cusps.

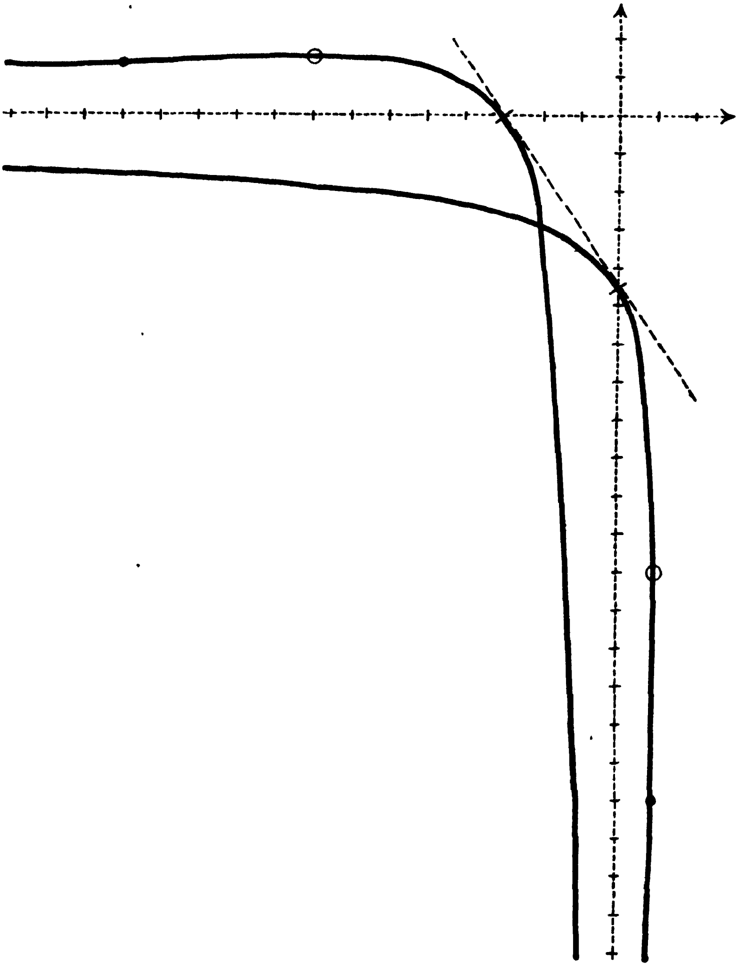

Fig. 1.

If we write down the condition that (i) considered as an equation in t has equal roots, we get an equation giving the parameter T ($\neq t$) of a point P on the curve from which two coincident tangents can be drawn not coinciding with the tangent at P.

It is clear that P is either an intersection of the curve with an inflexional tangent, or is a node. In the latter case the coincident tangents from P are the tangent to the other branch of the curve through the node.

If the curve is referred to Cartesian axes, the coordinates of any point being

$$x = f(t)/\psi(t), \quad y = \phi(t)/\psi(t),$$

the results of §§ 1, 2 evidently still hold with slight modifications; for instance, the tangent at any point is obtained by putting 1 for z in equation (ii) of § 1.

Ex. 1. Find the double points, inflexions, bitangent, degree, and class of

$$x = (2t-1)\,a/t^2, \qquad y = -6\,(t^2+t)\,a.$$

[The curve meets $\lambda x + \mu y + a = 0$ where

$$(2t-1)\,\lambda - 6\,(t^4+t^3)\,\mu + t^2 = 0.$$

This equation is of the fourth degree in t, so that the curve is a quartic.

Any tangent to the curve is

$$3t^3\,(2t+1)\,x - (t-1)\,y - 9t\,(2t^2-1)\,a = 0.$$

This equation is of the fourth degree in t, so that the curve is of the fourth class.

The equation (v) of § 1 giving cusps and inflexions is

$$t^2\,(2t^2-2t-1) = 0.$$

Since the factor t occurs twice on the left-hand side, $t=0$ gives a cusp. If in the original values of x and y we replace t by $1/T$ and repeat the process, we see that $T=0$, i.e. $t=\infty$, also gives a cusp. The inflexions are given by

$$2t^2-2t-1=0 \qquad \text{or} \qquad t = \tfrac{1}{2}(1 \pm \sqrt{3}).$$

The equation of the line joining the points with parameters t and T is

$$6t^2T^2\,(t+T+1)\,x + (t+T-2tT)\,y \\ + 6a\,\{(1-2tT)\,(t^2+tT+T^2) + (t+T)\,(t-T)^2\} = 0.$$

The coefficients all vanish if $t+T=-1$, $2tT=-1$. This gives $t = \tfrac{1}{2}(-1 \pm \sqrt{3})$ as the parameters of the node.

The tangents at the points with parameters t, T meet at the point

$$\frac{3x}{-2u^2+2uv+2v+1} = \frac{y}{v\,(2u^2-4v^2+u-2v)} \\ = \frac{9a}{-2u^3+2u^2v-u^2+5uv-2v^2+v},$$

where $u = t+T$, $v = tT$.

If the tangents coincide in a bitangent, the denominators of these fractions vanish. Eliminating v we get

$$u\,(2u+1)\,(u^2-2u-2) = 0,$$

and obtain $u = -\tfrac{1}{2}$, $v = -\tfrac{1}{2}$ as the values of u and v required. Hence $t=-1$ and $\tfrac{1}{2}$ give the parameters of the points of contact of the bitangent.

As an alternative method notice that the tangent at the point with parameter t meets the curve again at a point with parameter T where

$$2\,(t-1)\,T^2 + 2\,(t-1)\,(2t+1)\,T - t\,(2t+1) = 0 \quad . \quad . \quad . \quad \text{(ii)}.$$

This gives coincident values of T if $t = 1, \ -\frac{1}{2}, \ -1, \ \frac{1}{2}$; the corresponding values of T being $\infty, 0, \frac{1}{2}, -1$. Hence $1, -\frac{1}{2}$ are the parameters of the points of contact of tangents from the cusps; and $-1, \frac{1}{2}$ are the parameters of the points of contact of the bitangent, as before obtained.

Again, the equation (ii) considered as an equation in t has equal roots if $2T^2+2T-1 = 0$, which gives the parameters of the node; or $2T^2+10T-1 = 0$, which gives the parameters of the intersections of the curve with the two inflexional tangents.

The curve is (see Fig. 1)

$$x^2y^2+12a^3(3x+2y)+108a^4 = 0.]$$

Ex. 2. Find the double point and inflexions of

$$x = t^2+t+1, \qquad y = t^3+t^2+t+1.$$

[Acnode is (0, 1) given by $t^2+t+1 = 0$. Inflexions are (1, 1), (1, 0), $(0, \infty)$ given by $t = 0, -1, \infty$.]

Ex. 3. Find the double point and inflexions of

$$x = (t-1)^3, \qquad y = t^3, \qquad y = (t+1)^3.$$

[Acnode is (8, −1, 8) given by $3t^2+1 = 0$. Inflexions are (0, 1, 8), (1, 0, −1), (8, 1, 0) given by $t = 1, 0, -1$.]

Ex. 4. Find the double point and inflexions of

$$x = t^2+1, \qquad y = t^3+3t+4, \qquad z = t^2-t.$$

[Crunode is (1, 4, 0) given by $t = 0, 1$. Inflexions given by

$$(2^{\frac{1}{3}}-1)t+1 = 0, \text{ \&c.}]$$

Ex. 5. Find the double point and inflexions of

$$x = t^2+1, \qquad y = t^3+3t+4, \qquad z = t^2+t.$$

[Acnode is (3, 8, 4) given by $t^2-3t+4 = 0$. Inflexions given by

$$t^3-3t^2-3t+7 = 0.]$$

Ex. 6. Find the double point and inflexions of

$$x = \cos 3\phi, \qquad y = \sin 3\phi, \qquad z = \cos\phi.$$

[Crunode is (2, 0, −1) given by $\phi = \pm\frac{1}{3}\pi$. Inflexions are (0, 1, 0) and $(1, \pm i, 0)$.]

Ex. 7. Find the double points, inflexions, and bitangents of

$$x = t^4, \qquad y = 1+t^2, \qquad z = t.$$

[Acnodes are (−1, 1, 1) given by $t^2-t+1 = 0$ and (1, −1, 1) given by $t^2+t+1 = 0$. Cusp is (1, 0, 0) given by $t = \infty$.

Inflexions are given by $t = 0, 0, \pm\sqrt{2}$; the coincident values of t implying that $t = 0$ gives the point of undulation (0, 1, 0).

The bitangents are $x = 0$ and $x+4y = 0$ given by $t^2 = 0$ and $2t^2+1 = 0$.]

Ex. 8. Find the inflexions of

$$x = t^3-137, \qquad y = t^4+255, \qquad z = t-9.$$

[$t = \infty, 0, -1, -3, 5, 17$.]

Ex. 9. Find the double points, inflexions, and bitangents of

$$x = a(t^4-2t^2), \qquad y = a(t^3-3t).$$

[Cusps where $t^2 = 1$, crunode where $t^2 = 3$, point of undulation with infinitely distant tangent where $t = \infty$.]

Ex. 10. Find the double points and inflexions of

$$x = a(t^4+8t), \qquad y = a(t^3+1).$$

[Triple point where $t^3+8=0$, inflexions where $t(t^3-4)=0$, undulation where $t=\infty$.]

Ex. 11. Find the double points and inflexions of

$$x : y : z = (t^4+at^3) : (t+b) : t^2.$$

[There are cusps at $(1, 0, 0)$ and $(0, 1, 0)$ given by $t=\infty$ and $t=0$. The third double point is also a cusp if $a=4b$, given by $t=-2b$.]

Ex. 12. Find the double points and bitangent of

$$x : y : z = (t^2-1) : 2t^3 : (t^2+1)^2.$$

[Cusps where $t=\pm\sqrt{3}, \infty$. Bitangent $z=0$, the points of contact given by $t=\pm i$. Any tangent is

$$t(t^2+1)x-(t^2+1)y+tz=0.]$$

Ex. 13. Find the double points and inflexions of

$$x : y : z = (1+t^2) : -1 : (1+t+2t^2+t^4).$$

[Double point at $t=\infty$, inflexions at $t=\frac{1}{2}, \frac{1}{2}\omega, \frac{1}{2}\omega^2$.]

Ex. 14. If $x : y : z = f(t) : \phi(t) : \psi(t)$ is a curve and

$$f(\alpha+i\beta) \equiv f_1(\alpha, \beta^2)+i\beta f_2(\alpha, \beta^2), \text{ \&c.},$$

the acnodes are given by solving for α and β^2 from

$$\phi_1\psi_2=\phi_2\psi_1, \qquad \psi_1 f_2=\psi_2 f_1, \qquad f_1\phi_2=f_2\phi_1.$$

Apply the method to Ex. 7.

Ex. 15. The equation of the conic of closest contact at any point of

$$x : y : z = f(t) : \phi(t) : \psi(t)$$

is obtained by equating to zero the determinant whose first row has the elements x^2, y^2, z^2, yz, zx, xy, whose second row has the elements f^2, ϕ^2, ψ^2, $\phi\psi$, ψf, $f\phi$, and whose other four rows have as elements the first, second, third, and fourth derivatives of these with respect to t.

Find similarly the cubic, quartic, ... of closest contact.

[The reader may apply the same method to find the osculating sphere at any point of a twisted curve, &c.

The evaluation of the determinant when f, ϕ, ψ are polynomials is easily carried out by noticing that the determinant, whose first row is

$$t^{\alpha_1}, \quad t^{\alpha_2}, \quad \ldots, \quad t^{\alpha_n}$$

and whose other rows are the 1st, 2nd, ..., $(n-1)$-th derivatives of these with respect to t, is Rt^a where

$$R \equiv (\alpha_n-\alpha_{n-1})(\alpha_n-\alpha_{n-2})\ldots(\alpha_n-\alpha_1)(\alpha_{n-1}-\alpha_{n-2})\ldots(\alpha_2-\alpha_1),$$
$$a \equiv \alpha_1+\alpha_2+\ \ldots\ +\alpha_n-\tfrac{1}{2}n(n-1).$$

If f, ϕ, ψ are trigonometric or hyperbolic functions, we may express them in terms of exponentials, using the fact that if the first row of the determinant is

$$e^{\alpha_1 t+\beta_1}, \quad e^{\alpha_2 t+\beta_2}, \quad \ldots, \quad e^{\alpha_n t+\beta_n},$$

its value is Re^b, where

$$b=(\alpha_1+\alpha_2+\ \ldots\ +\alpha_n)t+(\beta_1+\beta_2+\ \ldots\ +\beta_n).]$$

Ex. 16. Find the conic of closest contact at any point of

$$\text{(i)}\ x : y : z = t^p : t^q : 1.$$
$$\text{(ii)}\ x : y : z = t : t^2 : 1 + t^3.$$

[(i) $(p+q)(p-2q)q^2t^{2q}x^2 + (p+q)(q-2p)p^2t^{2p}y^2$
$+ (p-q)^2(p-2q)(q-2p)t^{2p+2q}z^2 + 4(p+q)(p-q)(p-2q)pt^{2p+q}yz$
$+ 4(p+q)(q-p)(q-2p)qt^{p+2q}zx + 4(p-2q)(q-2p)pqt^{p+q}xy = 0.$

(ii) $(5t^9 + 5t^6 + 10t^3 + 1)x^2 + t(t^9 + 10t^6 + 5t^3 + 5)y^2 + t^5z^2$
$- (5t^6 + 1)yz - t^4(t^6 + 5)zx - 2t^2(5t^6 + t^3 + 5)xy = 0.$]

Ex. 17. The parameters of the sextactic points (at which a conic has six-point contact) of the curve of Ex. 15 are given by equating to zero the determinant whose first row is

$$f^2,\quad \phi^2,\quad \psi^2,\quad \phi\psi,\quad \psi f,\quad f\phi$$

and whose other rows are the first, second, third, fourth, and fifth derivatives of these.

Ex. 18. There is no non-degenerate conic having six-point contact with $y^pz^q = x^{p+q}$.

Ex. 19. The non-degenerate conics of closest contact with $a^{q-p}y^p = x^q$ are all ellipses if $(p-2q)(q-2p) > 0$, and all hyperbolas if

$$(p-2q)(q-2p) < 0.$$

[Use Ex. 16 (i).]

Ex. 20. The non-degenerate conics of closest contact with $y^pz^q = x^{p+q}$, where p and q are positive integers, meet $x = 0$ in real points. Their intersections with $z = 0$ are real if $q \geqslant p$, otherwise unreal.

[Replace q by $p+q$ in Ex. 16 (i).]

§ 3. Deficiency not Negative.

Suppose now that a curve of degree n has deficiency D, so that

$$\delta + \kappa = \tfrac{1}{2}(n-1)(n-2) - D;$$

and suppose that the curve does not degenerate into two or more curves of lower degree.

A curve of degree $n-2$ can be drawn through the

$$\tfrac{1}{2}(n-1)(n-2) - D$$

double points and through any

$$\tfrac{1}{2}(n-2)(n+1) - \tfrac{1}{2}(n-1)(n-2) + D = n - 2 + D$$

other fixed points of the given n-ic, since the $(n-2)$-ic is determined by $\tfrac{1}{2}(n-2)(n+1)$ points. (Ch. II, § 6).

The $(n-2)$-ic and n-ic meet twice at each double point of the n-ic and once at each of the $n-2+D$ other fixed points. They therefore meet at

$$n(n-2) - 2\{\tfrac{1}{2}(n-1)(n-2) - D\} - \{n - 2 + D\} = D$$

other points, since the n-ic and $(n-2)$-ic meet in $n(n-2)$ points.*

It follows that:

The deficiency of a non-degenerate curve cannot be negative.

We have so far supposed that the n-ic has only ordinary nodes and cusps. Now let it have ordinary multiple points. An $(n-2)$-ic can be drawn with a $(k-1)$-ple point at each k-ple point of the n-ic, and also passing through $n-2+D$ other fixed points of the n-ic. For by Ch. VIII, § 3

$$D = \tfrac{1}{2}(n-1)(n-2) - \Sigma \tfrac{1}{2}k(k-1),\dagger$$

and by Ch. II, § 6 we have subjected the $(n-2)$-ic to

$$\Sigma \tfrac{1}{2}k(k-1) + (n-2) + D = \tfrac{1}{2}(n-2)(n+1)$$

conditions. But the n-ic and $(n-2)$-ic meet in

$$n(n-2) - \Sigma k(k-1) - (n-2+D) = D$$

other points, so that the previous argument still applies.

It should be noticed, however, that there may be other restrictions on the nature of the multiple points of a non-degenerate curve in addition to $D \geqslant 0$ (see Ex. 1, 2, 3 and Ch. IV, § 7, Ex. 5 to 7; Ch. XVII, § 6, Ex. 1; &c.).

If the curve has higher singularities, it may be transformed as in Ch. IX into a curve with ordinary multiple points without altering the deficiency. Hence in this case also $D \geqslant 0$.

Ex. 1. An n-ic cannot have two multiple points of orders k_1 and k_2, if $k_1 + k_2 > n$.

[For otherwise the line joining the points would meet the curve in more than n points.]

Ex. 2. An n-ic cannot have multiple points of orders k_1, k_2, k_3, k_4, k_5, if $k_1 + k_2 + k_3 + k_4 + k_5 > 2n$.

[Consider the intersections of the n-ic with the conic through the points.]

Ex. 3. If a non-degenerate n-ic has $\tfrac{1}{2}p(p+3)$ triple points,

$$n \geqslant \tfrac{3}{2}(p+3).$$

[Consider the intersections of the n-ic with a p-ic through the triple points.]

Ex. 4. If an n-ic degenerates into r curves of deficiencies

$$D_1, D_2, \ldots, D_r,$$

it has $\tfrac{1}{2}n(n-1) - (n-r) - (D_1 + D_2 + \ldots + D_r)$ double points.

[Each intersection of two constituents of the n-ic counts as a node of the n-ic.]

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* If the n-ic was degenerate, the $(n-2)$-ic might be part of the n-ic, and this statement would not be correct.

† The summation is extended over all the multiple points of the n-ic.

Ex. 5. If a degenerate n-ic has $\frac{1}{2}n(n-1)$ nodes, it consists of n straight lines. If it has $\frac{1}{2}n(n-1)-1$ nodes, it consists of $n-2$ straight lines and a conic. If it has $\frac{1}{2}n(n-1)-2$ nodes, it consists of $n-4$ straight lines and two conics.

[See Ex. 4.]

§ 4. Unicursal Curves.

Consider now an n-ic $f=0$ with zero deficiency. Suppose that a variable N-ic is subjected to certain conditions, such as passing through fixed points of the n-ic or having assigned singularities at fixed points of the n-ic; which conditions are in all equivalent to $\frac{1}{2}N(N+3)-1$ independent linear relations between the coefficients of the equation of the N-ic. Suppose also that the N-ic meets the n-ic $nN-1$ times at these fixed points. We assume for the present that such a variable N-ic exists. Since the $\frac{1}{2}N(N+3)$ ratios of the coefficients in the equation of the N-ic are subjected to $\frac{1}{2}N(N+3)-1$ linear relations, the coefficients can be expressed linearly in terms of a single quantity t. Thus the equation of the N-ic is of the form $u+tv=0$, where $u=0$ and $v=0$ are two such N-ics; i. e. the N-ics form a pencil.

If Cartesian coordinates are used (a similar process holds good for homogeneous coordinates), we may eliminate y between $f=0$ and $u+tv=0$, obtaining an equation of degree nN in x with coefficients rational in t. Of the roots of this equation $nN-1$ are abscissae of fixed points, and the remaining root is the abscissa of the other intersection P of the n-ic and N-ic.

Dividing the equation by the factors corresponding to the abscissae of the fixed points, we have an equation of the first degree in x with coefficients rational in t, giving the abscissa of P.

In a similar manner the ordinate of P is expressed rationally in terms of t. But, by choosing t properly, P may be made any point of the n-ic. Hence:

The coordinates of any point on a curve of zero deficiency can be expressed as rational algebraic functions of a parameter.

This implies that, if the curve of § 1 is of zero deficiency, $f(t)$, $\phi(t)$, and $\psi(t)$ may be taken as polynomials in t.

A curve of zero deficiency is often called a *rational* or *unicursal* curve. The former name comes from the fact that the coordinates are expressed *rationally* in terms of a parameter.

The latter name comes from the fact that the curve can be drawn by a pen which never leaves the plane of the paper except to pass from one end of an asymptote to the other, or to insert the acnodes; in other words, the curve consists of a single circuit.* In fact, the coordinates of a point P on the curve being continuous functions of t, the point P moves continuously as t varies (provided P is finite). This would not hold if t were complex; but, if a real point P on the curve is given by $t = \alpha+\beta i$, where α and β are real, P is also given by $t = \alpha-\beta i$, i. e. P is a double point. Since any value of t close to $\alpha+\beta i$ gives evidently an unreal point of the curve, there is no real part of the curve in the neighbourhood of P; i. e. P is an acnode.

We now show that there always exists an N-ic with the properties stated at the beginning of this section.

First suppose the n-ic has no multiple points other than ordinary nodes and cusps. We take $N = n-2$, and subject the N-ic to the conditions that it shall pass through each of the $\frac{1}{2}(n-1)(n-2)$ nodes and cusps of the n-ic and also pass through $n-3$ other fixed points † of the n-ic. Then the coefficients in the equation of the N-ic are connected by

$$\tfrac{1}{2}(n-1)(n-2)+(n-3), \text{ i. e. } \tfrac{1}{2}N(N+3)-1$$

linear relations; and the N-ic meets the n-ic

$$2\times\tfrac{1}{2}(n-1)(n-2)+(n-3), \text{ i. e. } nN-1$$

times at the fixed points; as required.

Next suppose the n-ic has ordinary multiple points as well as ordinary nodes and cusps. We take $N = n-2$, and subject the N-ic to the conditions that it shall have a $(k-1)$-ple point at each k-ple point of the n-ic, and also pass through $n-3$ other fixed points of the n-ic. Then the coefficients in the equation of the N-ic by Ch. II, § 6 are connected by

$$\Sigma\tfrac{1}{2}k(k-1)+n-3, \text{ i. e. } \tfrac{1}{2}N(N+3)-1$$

linear relations, and the N-ic meets the n-ic

$$\Sigma k(k-1)+n-3, \text{ i. e. } nN-1$$

times at the fixed points; as required.

For by Ch. VIII, § 3

$$D = \tfrac{1}{2}(n-1)(n-2)-\Sigma\tfrac{1}{2}k(k-1) = 0.$$

Lastly suppose that the n-ic has higher singularities. We take the N-ic as an adjoined $(n-2)$-ic passing through $n-3$ fixed ordinary points of the n-ic, and use the result of Ch. IX, § 10. Or as an alternative we may transform the n-ic by

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* See Ch. XX, §§ 1, 9. Of course, curves of deficiency other than zero may consist of a single circuit; e. g. $y^2x+(x-1)(x^2+2x+2) = 0$.

† It is often convenient to take them as points on the n-ic consecutive to double points.

Ch. IX into a curve with only ordinary multiple points, and apply the above process to the transformed curve.

But in the case of an n-ic whose singularities are not all ordinary cusps and nodes it is often possible to take for N a value less than $n-2$, thus saving much labour in the practical applications (see Ex. 5, 6).

We now show that, conversely,

If the coordinates of any point on a curve can be expressed rationally and algebraically in terms of a parameter, the curve is of zero deficiency.

For suppose that in § 1 $f(t)$, $\phi(t)$, $\psi(t)$ are polynomials of degree n, the curve being therefore of degree n. The tangent at any point is, by § 1,

$$\begin{vmatrix} x & y & z \\ nf-f' & n\phi-\phi' & n\psi-\psi' \\ f' & \phi' & \psi' \end{vmatrix} = 0.$$

If we consider this as an equation in t, (x, y, z being taken as constant), it is evidently of degree not greater than $2n-2$. Its solutions will include the parameters of the κ cusps for which $f'/f = \phi'/\phi = \psi'/\psi$ (§ 1), and the parameters of the m tangents which can be drawn from (x, y, z) to the curve. Hence $m+\kappa \leqslant 2n-2$. But the class m is given by

$$m = n(n-1)-2\delta-3\kappa = 2n-2-\kappa+2D;$$

so that $m+\kappa = 2n-2+2D$.

Now D is not negative, so that we must have $m+\kappa = 2n-2$ and $D=0$, as stated.

It should be noted that the parameters of three assigned points on a unicursal curve may be taken as any three assigned quantities, say T_1, T_2, T_3. For suppose that, when the coordinates of any point on the curve are expressed rationally in terms of the parameter t, the parameters of the three points are t_1, t_2, t_3. Then we have only to express the coordinates in terms of the parameter $T = (\alpha t+\beta)/(\gamma t+\delta)$; where α, β, γ, δ are constants chosen to satisfy

$$T_1 = (\alpha t_1+\beta)/(\gamma t_1+\delta), \quad T_2 = (\alpha t_2+\beta)/(\gamma t_2+\delta),$$
$$T_3 = (\alpha t_3+\beta)/(\gamma t_3+\delta).$$

Ex. 1. Express the coordinates of any point on the curve

$$4(x-1)^3+(y-3x+2)^2 = 0$$

rationally in terms of a parameter.

[The line $y-1 = t(x-1)$ through the cusp (1, 1) meets the curve where $x = 1+\frac{1}{4}(t-3)^2$, $y = 1+\frac{1}{4}t(t-3)^2$.]

Ex. 2. Express rationally in terms of a parameter the coordinates of any point on the curves

(i) $x^3+y^3=3axy.$

(ii) $(2x^2+2y^2+1)(y-x)=x^2-6xy+y^2.$

(iii) $x^3+y^3+4xz(x+y)=0.$

(iv) $(x+y)(x-y)^2=4xyz.$

Ex. 3. Express the coordinates of any point on the curve

$$(x^2-y^2)^2-(x^2-y^2)(x+3y)-3x^2+6xy+y^2=0$$

rationally in terms of a parameter.

[A variable conic through the double points

$$(0,0),\qquad x=-y=\infty,\qquad x=y=\infty$$

and one other fixed point R on the curve meets this quartic in one more point. Choose R as the point on the curve consecutive to the cusp $x=y=\infty$. The conic is then

$$(x+y-t)(x-y-1)=t$$

meeting the quartic where

$$x=\frac{(t^2+2t-2)(t^2+t-1)}{2(t^2+t-2)},\qquad y=\frac{(t^2+2t-2)(t^2+t-3)}{2(t^2+t-2)}.$$

See § 6, Ex. 1 and Ch. III, Fig. 3.]

Ex. 4. Express rationally in terms of a parameter the coordinates of any point on the curves

(i) $x^2y^2+2x^2y+xy^2-3x^2+2xy-2y^2=0.$

(ii) $2y^2(2x^2-5x+4)-xy(x-2)(x-8)+2x^2(x-2)^2=0.$

(iii) $x^2y^2+12a^3(3x+2y)+108a^4=0.$

(iv) $xy(2xy-5x-10y)+50(x-y)^2=0.$

(v) $(x^2-y^2)^2+(x^2-y^2)(5y-x)-3x^2-2xy+9y^2=0.$

(vi) $y^2z^2+z^2x^2+x^2y^2=2xyz(x+y+z).$

(vii) $2ay^3-3a^2y^2=x^4-2a^2x^2.$

(viii) $r=a(1+\cos\theta).$

(ix) $r^2=a^2\cos 2\theta.$

(x) $(y+x^2)^2=4x^2(x+k).$

[(i) Meets $y(1-x)=tx$ where

$$(t+1)(3-t)x=2t^2-2t+3=(4-3t)y.$$

(ii) Meets $y(2+tx)+x^2-2x=0$ where

$$x=8\div(2t^2+t+4),\qquad y=8t(2t+1)\div(2t^2+t+4)(2t^2+5t+4).$$

(iii) Meets $x(y+6at)+6a^2(2t-1)=0$ where

$$x=(2t-1)a\div t^2,\qquad y=-6(t^2+t)a.$$

(iv) Meets $txy=5(x-y)$ where

$$x=15\div(2-t+2t^2),\qquad y=15\div2(1+t+t^2).$$

(v) Meets $(x-y+1)(x+y+t)=t$ where

$$x=-(t^2-6t+2)(t^2-3t+5)\div6(t-1)(t-2),$$
$$y=-(t^2-6t+2)(t^2-3t-1)\div6(t-1)(t-2).$$

(vi) Meets $z(x-y)=txy$ where

$$x(t-1)^2=y(t+1)^2=4z.$$

(vii) Meets $x^2-ay = tx(y-a)$ where

$$x = (2-3t^2)a \div t^3, \qquad y = (2-3t^2)(2-t^2)a \div 2t^4.$$

(viii) Meets $ay = t(x^2+y^2)$ where

$$x = 2(1-t^2)a \div (1+t^2)^2, \qquad y = 4ta \div (1+t^2)^2.$$

(ix) Meets $x^2+y^2 = at(x-y)$ where

$$x = t(t^2+1)a \div (t^4+1), \qquad y = t(t^2-1)a \div (t^4+1).$$

(x) Meets $y+x^2 = 2tx$ where

$$x = t^2-k, \qquad y = (t^2-k)(k+2t-t^2).]$$

Ex. 5. Express the coordinates of any point on

$$(x+y)^3 z^2 = x^3 (y+z)^2$$

rationally in terms of a parameter.

[This quintic has a triple point at (0, 0, 1), a node at (1, 0, 0), and cusps at (0, 1, 0) and (1, −1, 1). In the case of a 5-ic with a triple point and three double points the N-ic of § 4 may be taken as a conic through these multiple points. In fact $x(z+y) = tz(x+y)$ meets the curve again where $(t^2-1)x = y = (t^3-1)z$.

We may illustrate the general theory for which $N = n-2$ by taking the N-ic as the cubic

$$xyz + y^2z + t(x^2y + 2x^2z - y^2z) = 0;$$

which has a double point at (0, 0, 1), passes through (1, 0, 0), (0, 1, 0), (1, −1, 1) and through two more fixed points on the 5-ic consecutive to (1, 0, 0) and (0, 1, 0) respectively. It meets the 5-ic again where

$$t(3t-2)(t-1)x = (t-1)^3 y = t(7t^2-9t+3)z.$$

An alternative method is to transform the curve by quadratic transformation, replacing $x : y : z$ by $1/x : 1/y : 1/z$, and to apply the method of § 4 to the transformed curve, which is a cuspidal cubic. The reader may apply this process to Ex. 4 (i), (iv), (vi).

Yet another method is given in § 6, Ex. 3.]

Ex. 6. Express rationally in terms of a parameter the coordinates of any point on the curves

(i) $x^4+y^4 = ax(x^2-xy+y^2)$.

(ii) $x(x+2)^4 = y^4(1-y)$.

(iii) $x^3(y+z)^3 + y^3(x+z)^3 = x^2y^2(x+z)(y+z)$.

(iv) $y^3z^2 + y^2z^3 + z^3x^2 + z^2x^3 + x^3y^2 + x^2y^3$
$$= 2xyz(2x^2+2y^2+2z^2-yz-zx-xy).$$

(v) $(x^2+y)(x^2-2y) = x^3y$,

(vi) $(y-x^2)^2 = x^5+x^3y$.

(vii) $(y+x^3)(y+2x^3)(y+3x^3) = y^2x^4$.

(viii) $zu_{n-1}+u_n = 0$.

(ix) $x(xy+az^2)^2 + y(xy+bz^2)^2 = 0$.

(x) $x(xy+az^2)^2 + y(xy+bz^2)^2 + z(xy+az^2)(xy+bz^2) = 0$.

(xi) $(yz+x^2) = y^3(x+y)$.

[Find the intersections of the curve with (i) $y = tx$, (ii) $y = t(x+2)$, (iii) $ty(x+z) = x(y+z)$, (iv) $ty(x-z) = x(y-z)$, (v) $y = tx^2$ (see Ch. III, Fig. 10), (vi) $y = tx^2$, (vii) $y = tx^3$, (viii) $y = tx$, (ix) $xy = -t^2z^2$, (x) $4xy = (1-t^2)z^2$, (xi) $y = (t^2-1)x$.]

Ex. 7. If in § 4 the n-ic has only ordinary cusps and nodes, N must be $n-1$ or $n-2$.

[Suppose the fixed intersections of the n-ic and N-ic are a double points and b other points of the n-ic. Then

$$a+b = \tfrac{1}{2}N(N+3)-1, \qquad 2a+b = nN-1,$$

giving $$a = \tfrac{1}{2}N(2n-N-3), \qquad b = N(N+3-n)-1.$$

Hence $N > n-3$. See Clebsch, *Crelle*, lxiv (1865), p. 44.]

Ex. 8. Three points on a curve have parameters t_1, t_2, t_3. Transform the parameter so that the points have new parameters $0, 1, \infty$.

[The new parameter is $(t-t_1)(t_2-t_3)/(t-t_3)(t_2-t_1)$.]

Ex. 9. In the n-ic $x:y:z=f(t):\phi(t):\psi(t)$, where f, ϕ, ψ are polynomials of degree n with the coefficients of t^{n-1} zero, the sum of the parameters of the intersections of the n-ic with any algebraic curve is zero.

Ex. 10. The curve of Ex. 9 has a cusp. Conversely, any unicursal curve with a cusp can be put in this form.

[The cusp is given by $t=\infty$. Conversely, if the curve has a cusp given by $t=\alpha$, take as a new parameter $T \equiv \beta + 1/(t-\alpha)$, where β is chosen to make zero the coefficient of T^{n-1} in x.]

Ex. 11. Find the Plücker's numbers of $x:y:z=f(t):\phi(t):\psi(t)$; f, ϕ, ψ being polynomials in t.

[n and m are given by §§ 1 and 2, while $D=0$.]

Ex. 12. Find the Plucker's numbers of $y^p z^q = x^{p+q}$.

[The tangent at $(t^p, t^{p+q}, 1)$ is $(p+q)t^q x = py + qt^{p+q}z$. Hence

$$D=0, \qquad n=m=p+q, \qquad \delta=\tau=\tfrac{1}{2}(p+q-2)(p+q-3),$$
$$\kappa=\iota=p+q-2.$$

See also Ch. VIII, § 5, Ex. 1.

For other examples find the Plücker's numbers of the examples in Ch. III, § 9.]

Ex. 13. In the curve $x:y:z=f(t):\phi(t):\psi(t)$, where f, ϕ, ψ are polynomials, to each point on the curve corresponds *in general* a single value of t; but this is not universally the case. Suppose that to each point on the curve corresponds r values of t, and that the highest common factor of

$$f(t)\,\psi(t_1)-f(t_1)\,\psi(t) \quad \text{and} \quad \phi(t)\,\psi(t_1)-\phi(t_1)\,\psi(t)$$

is arranged in powers of t_1, the coefficients being polynomials in t. Then, if T is the ratio of the coefficients of any two powers of t_1 and is not a constant, $x:y:z$ can be expressed rationally in terms of the parameter T, so that to each point on the curve corresponds a single value of T.

[If to a point on the curve correspond the values $t_1, t_2, \ldots, t_r$ of t, the highest common factor is $(t-t_1)(t-t_2)\ldots(t-t_r)$. The coefficients of any two powers of t in this are symmetric functions of $t_1, t_2, \ldots, t_r$; and therefore their ratio has a single value for each point on the curve. But the expressions whose highest common factor was taken are only changed in sign by interchange of t and t_1. See Lüroth, *Math. Annalen*, ix, p. 163.]

Ex. 14. In the following curves express the coordinates in terms of a parameter so that to each point of the curve corresponds only a single value of the parameter.

(i) $x : y : z = t^2 : 1 : t^6$.
(ii) $x : y : z = (t^2+1)^2 : t\,(t^2+1) : (t^4+3t^2+1)$.
(iii) $x : y : z = t\,(t^4+1) : (t^6-1) : t^3$.
(iv) $x : y : z = t\,(t-1)\,(t^2+t-1) : (t^3+1)^2 : t^2$.

[The H.C.F. of Ex. 13 is
(i) $t^2-t_1^2$. If $T=t^2$, $x : y : z = T : 1 : T^3$.
(ii) $t^2t_1-t\,(t_1^2+1)+t_1$. If $T=t+t^{-1}$, $x : y : z = T^2 : T : (T^2+1)$.
(iii) $t^2t_1+t\,(1-t_1^2)-t_1$. If $T=t-t^{-1}$, $x : y : z = (T^2+2) : (T^3+3T) : 1$.
(iv) $t^3t_1-t\,(1+t_1^3)\,t+t_1$. If $T=t^2+t^{-1}$, $x : y : z = (T-2) : T^2 : 1$.]

Ex. 15. The homogeneous equation of $x : y : z = f(t) : \phi(t) : \psi(t)$ may be derived by equating to zero the coefficients of $1, u, u^2, u^3, \ldots$ in

$$\frac{1}{t-u}\begin{vmatrix} x & y & z \\ f(t) & \phi(t) & \psi(t) \\ f(u) & \phi(u) & \psi(u) \end{vmatrix}.$$

and eliminating $1, t, t^2, t^3, \ldots$ between the equations so obtained.

[We thus get the equation by equating to zero a determinant of order n, if f, ϕ, ψ are polynomials of degree n. See Richmond, *Bull. Amer. Math. Soc.*, xxiii (1916), p. 90.]

Ex. 16. The line joining corresponding points (with the same parameter) of a unicursal n-ic and a unicursal N-ic envelops a curve of class $n+N$.

Ex. 17. If more than $n_1+n_2+n_3$ trios of corresponding points on given unicursal curves of degree n_1, n_2, n_3 are collinear, every trio of corresponding points is collinear.

Ex. 18. If a, b are the parameters of a given node on a given n-ic

$$x : y : z = f(t) : \phi(t) : \psi(t),$$

where f, ϕ, ψ are polynomials, and $F(t) = 0$ is the equation giving the parameters of the nr intersections of any r-ic with the n-ic, then $F(a)/F(b)$ is the same whatever r-ic is chosen.

[If the r-ic is $\Sigma A x^{\alpha} y^{\beta} z^{\gamma} = 0$, where $\alpha+\beta+\gamma = r$, then

$$F(t) \equiv \Sigma A f^{\alpha} \phi^{\beta} \psi^{\gamma};$$

and $\quad F(a)/F(b) = \{f(a)/f(b)\}^r = \{\phi(a)/\phi(b)\}^r = \{\psi(a)/\psi(b)\}^r$.]

Ex. 19. Apply § 1 (v) and the relation $2\kappa+\iota = 3\,(n-2)+6D$ to show that, if the coordinates of any point on a curve can be expressed rationally in terms of a parameter, $D = 0$.

[The degree of $F(t)$ in t is $2\kappa+\iota$.]

Ex. 20. A line AB of length c slides with its ends on the rectangular axes OX, OY. Find the locus of the point of contact with AB of the circle inscribed in the triangle ABO.

[$x/c = t\,(1-t)\,(1+t)^2/(1+t^2)^2$, $\quad y/c = 2t\,(1-t)/(1+t^2)^2$, where t is the tangent of half the angle between AB and OX. Hence the locus is a unicursal quartic with a node at the origin.]

Ex. 21. If A is an end of an axis and P any point on an ellipse, find the loci of the intersection of the line through A perpendicular to AP with the tangent and normal at P.

[A unicursal cubic and quartic.]

Ex. 22. A triangle ABC of fixed size and shape turns about C. Show that the locus of the intersection of HB and KC is a unicursal quartic, H and K being fixed points.

Ex. 23. The locus of the poles of any normal to a given conic is in general a unicursal quartic. Consider the case in which the conic is a parabola.

Ex. 24. A range of points on a conic is homographic with a pencil of lines. Any line of the pencil meets the tangent at the corresponding point of the conic on a fixed unicursal cubic.

[Its node is at the vertex of the pencil.]

Ex. 25. The locus of the intersection of the tangents at corresponding points of homographic ranges on two given conics is a unicursal quartic.

[The tangents can be put in the form

$$Pt^2+2Qt+R=0, \qquad pt^2+2qt+r=0,$$

where P, Q, R, p, q, r are linear in x, y, z.]

§ 5. Coordinates of a Point in terms of Trigonometric or Hyperbolic Functions.

Instead of expressing the coordinates of any point on a unicursal curve rationally in terms of the parameter t, it may be more convenient to express them rationally in terms of $\cos\phi$ and $\sin\phi$ or of $\cosh\phi$ and $\sinh\phi$; where ϕ is a new parameter. The case of the ellipse is well known. Here the most convenient parameter is the eccentric angle of any point.

Putting $t=\tan\frac{\phi}{2}$, we can immediately change from the rational expression of the coordinates in terms of t to the rational expression in terms of $\cos\phi$ and $\sin\phi$; or conversely.

The following theorem is of interest:

The coordinates of any point on a unicursal n-ic $(n>2)$ *with an acnode can be expressed rationally in terms of* $\cos\phi$ *and* $\sin\phi$, *so that the parameters* $\phi_1, \phi_2, \ldots, \phi_{nr}$ *of the intersections of the n-ic with any r-ic satisfy the relation*

$$\phi_1+\phi_2+\ldots+\phi_{nr}\equiv 0 \ (mod.\ \pi).$$

For an n-ic with a crunode $\cos\phi$ *and* $\sin\phi$ *are replaced by* $\cosh\phi$ *and* $\sinh\phi$, *and* π *by* πi.

Suppose that the triangle of reference is taken so that $(1, 0, 0)$ is the acnode. We may suppose the parameters of the acnode to be i and $-i$ (see end of § 4). Then we have equations of the form

$$x = (a_0 t^n + a_1 t^{n-1} + \dots + a_n), \quad y = (t^2+1)(b_0 t^{n-2} + \dots + b_{n-2}),$$
$$z = (t^2+1)(c_0 t^{n-2} + \dots + c_{n-2}),$$

if (x, y, z) is any point of the n-ic.

First take $r = 1$. The n-ic meets any straight line

$$\lambda x + \mu y + \nu z = 0$$

where

$$\lambda(a_0 t^n + a_1 t^{n-1} + \dots + a_n)$$
$$+ (t^2+1)([b_0\mu + c_0\nu] t^{n-2} + [b_1\mu + c_1\nu] t^{n-3} + \dots) = 0.$$

If s_k is the sum of the products of the roots of this equation k at a time, we have at once

$$(s_1 - s_3 + s_5 - \dots) \div (1 - s_2 + s_4 - \dots) = (-a_1 + a_3 - a_5 + \dots)$$
$$\div (a_0 - a_2 + a_4 - \dots).$$

Hence, if we put $\tan\phi$ for t, $x : y : z$ are expressed rationally in terms of $\cos\phi$ and $\sin\phi$, so that, if any straight line meets the n-ic in the points for which $\phi = \phi_1, \phi_2, \dots, \phi_n$,

$$\tan(\phi_1 + \phi_2 + \dots + \phi_n) = (-a_1 + a_3 - a_5 + \dots) \div (a_0 - a_2 + a_4 - \dots),$$

or

$$\phi_1 + \phi_2 \dots + \phi_n \equiv \alpha \pmod{\pi},$$

where

$$\tan\alpha = (-a_1 + a_3 - a_5 + \dots) \div (a_0 - a_2 + a_4 - \dots).$$

Replacing ϕ by $\phi + \alpha/n$, we have the sum of the parameters of any n collinear points $\equiv 0 \pmod{\pi}$.

Similarly the sum of the parameters of the intersections of the n-ic with any r-ic is a constant (mod. π). That this constant is zero is evident by taking the r-ic as r straight lines.

If the point $(1, 0, 0)$ is a crunode, we take its parameters as 1 and -1, and put $t = \tanh\phi$.

For the corresponding result in the case of a cusp see § 4, Ex. 10.

§ 6.

If a variable N-ic $(N < n)$ passes through $\frac{1}{2}N(N+3) - 1$ fixed points of a unicursal n-ic (as in § 4) and meets the n-ic $nN - 2$ times at these fixed points, *two* of the family of N-ics touch the n-ic in general.* For the n-ic and N-ic meet at two more points. Let ϕ_1, ϕ_2 be their parameters and let β be the sum of the parameters of the $nN-2$ fixed intersections, the parameter of any point on the n-ic being taken as in § 5. Then

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* At a point not coinciding with one of the fixed points. One (or both) of the N-ics here obtained may meet the n-ic at a cusp instead of touching it. For $\phi_1 \equiv \phi_2$ in this case also.

$$\phi_1+\phi_2+\beta \equiv 0 \text{ (mod. } \pi).$$

If the N-ic touches the n-ic, $\phi_1 \equiv \phi_2$ and

$$\phi_1 \equiv -\tfrac{1}{2}\beta \text{ or } \equiv -\tfrac{1}{2}\beta+\tfrac{1}{2}\pi.$$

This result also follows from Ch. VII, § 10, Ex. 11 and 12, Ch. IX, § 10, Ex. 1 and 2.

The coefficients in the equation of the N-ic may be expressed linearly in terms of a parameter t. On eliminating y between the equations of the n-ic and N-ic we get an equation in x (or in $x:z$, if homogeneous coordinates are used). Dividing out by the factors corresponding to the $nN-2$ given intersections, we have a quadratic equation in x, whose coefficients are rational in t, giving the two variable intersections of n-ic and N-ic. Solving it we have

$$x = M \pm LX^{\frac{1}{2}}$$

where L, M are rational in t, and X is a polynomial in t.

Substituting either of these values of x in the equations of n-ic and N-ic, we have two equations for y which have a common solution of the form $M'+L'X^{\frac{1}{2}}$, where M' and L' are rational in t.

Since $X=0$ gives the two N-ics which touch the n-ic, X is of the second degree in t.

Hence the coordinates of any point on the n-ic may be expressed rationally in terms of t and an expression of the form $(at^2+2bt+c)^{\frac{1}{2}}$.

But t and $(at^2+2bt+c)^{\frac{1}{2}}$ can always be expressed rationally in terms of a new parameter T; for instance, by the substitution

$$at+b = 2(ac-b^2)^{\frac{1}{2}}T \div (T^2-1),$$

if a and $ac-b^2$ are positive; and similarly in other cases.

This process gives a method of expressing the coordinates of a point on a unicursal curve rationally in terms of a parameter alternative to that given in § 4; which, if less simple in theory, may be much easier in practice, since the value of N may be lower than in the method of § 4.

If the multiple points of the n-ic are all ordinary nodes or cusps, the N-ic may be taken as an $(n-3)$-ic through all the double points of the n-ic but two; since

$$\tfrac{1}{2}(n-3)\,n-1 = \tfrac{1}{2}(n-1)\,(n-2)-2,$$

$$2\{\tfrac{1}{2}(n-1)\,(n-2)-2\} = n\,(n-3)-2.$$

If the n-ic has a $(n-2)$-ple point, the N-ic may be taken as a straight line through the $(n-2)$-ple point; and so on.

Ex. 1. Express the coordinates of any point on the curve

$$(x^2-y^2)^2-(x^2-y^2)(x+3y)-3x^2+6xy+y^2=0$$

rationally in terms of a parameter.

[A variable line $x+y+\lambda=0$ through a node at infinity meets the curve where $2(\lambda-2)(\lambda+1)y=-\lambda^3+3\lambda+\lambda(3-\lambda)^{\frac{1}{2}}$.

Putting $\lambda=3-(t+1)^2$ we get the coordinates given in § 4, Ex. 3.

The second 'N-ic' of § 6 is the line at infinity: for this line passes through the infinite cusp.]

Ex. 2. Apply the method of § 6 to the examples in § 4, Ex. 4.

[As another example the reader may take the general conic or the quartic with nodes at the vertices of the triangle of reference.]

Ex. 3. Express the coordinates of any point on

$$(x+y)^3z^2=x^3(y+z)^2$$

rationally in terms of a parameter.

[$y=\lambda x$ meets the curve where $\lambda x/z=-1+(\lambda+1)^{\frac{1}{2}}$. Put $\lambda=t^2-1$, and we have $(t^2-1)x=y=(t^3-1)z$ as in § 4, Ex. 5.]

Ex. 4. In the argument of § 6 N must be $n-1$, $n-2$, or $n-3$ in the case in which all the multiple points of the n-ic are ordinary nodes or cusps.

[As in § 4, Ex. 7.]

§ 7. Curves with Unit Deficiency.

Consider now an n-ic with unit deficiency. Suppose that just as in § 4 a variable N-ic $u+tv=0$ is subjected to certain conditions which are in all equivalent to $\frac{1}{2}N(N+3)-1$ linear relations between the coefficients of the equation of the N-ic. But suppose that the N-ic meets the n-ic $nN-2$ times at the fixed points (not $nN-1$ times as in § 4). Then, when we eliminate y between the equations of the n-ic and N-ic, we get an equation in x $nN-2$ of whose roots are the abscissae of fixed points, while the remaining two roots are the abscissae of the two variable intersections P and Q of n-ic and N-ic. Dividing the equation by the factors corresponding to the abscissae of the fixed points we have an equation of the second degree in x with coefficients rational in t whose roots are the abscissae of P and Q. Suppose its solution is

$$x=M\pm LX^{\frac{1}{2}},$$

where M and L are rational in t and X is a polynomial in t. If we substitute either of these values of x in the equations of

n-ic and N-ic, we get two equations in y whose common root is also of the form

$$M' + L'X^{\frac{1}{2}},$$

where M' and L' are rational in t.

Now the values of t given by $X = 0$ are the parameters of the points of contact of those of the variable N-ics which touch the n-ic at a point other than one of the fixed points, and the parameters of the cusps of the n-ic which are not included among the fixed points. But by Ch. VII, § 10, Ex. 11 and 12 (cf. Ch. IX, § 10, Ex. 1), putting $p = 2$ and $D = 1$, we see that the number of such points of contact and cusps is four. Hence X is a polynomial of degree four in t. If then we assume the existence of the N-ic, we have:

The coordinates of any point on a curve of unit deficiency may be expressed rationally in terms of a parameter t and an expression X of the form

$$\{a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4\}^{\frac{1}{2}}.$$

It is well known that t and X can be simultaneously expressed as rational functions of the elliptic functions $sn\,u$, $cn\,u$, $dn\,u$; or if preferred, in terms of Weierstrass's elliptic function $\wp u$ and its derivative $\wp' u$, defined by

$$(\wp' u)^2 = 4(\wp u)^3 - g_2\wp u - g_3; \quad \mathop{\mathrm{L}'}_{u\to 0} u^2\wp u = 1.$$

For instance, if we take a as a constant defined by

$$a_0^2 . \wp a = -(a_0a_2 - a_1^2), \quad a_0^3 . \wp' a = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3,$$

where

$$a_0^2g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2, \qquad a_0^3g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix},$$

we have, on putting

$$t = -\frac{a_1}{a_0} + \frac{1}{2}\,\frac{\wp' u - \wp' a}{\wp u - \wp a},$$

$$\{a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4\}^{\frac{1}{2}} = a_0^{\frac{1}{2}}\{\wp u - \wp(u + a)\}.*$$

Hence:

The coordinates of any point on a curve of unit deficiency can be expressed rationally in terms of elliptic functions.

---

* If a_0 is negative, put $t = T^{-1} + h$ where $a_0h^4 + 4a_1h^3 + 6a_2h^2 + 4a_3h + a_4$ is positive, and proceed as above. See Halphen's *Fonctions elliptiques*, I, Ch. IV, p. 120.

The reader may also consult Picard, *Traité d'Analyse*, II, Ch. XV, §§ 12 and 16, Ch. XVII, §§ 2 and 4, or Goursat, *Cours d'Analyse*, II, §§ 340 and 362.

To prove the existence of the N-ic, we show that when the n-ic has only ordinary multiple points and cusps, an $(n-2)$-ic with a $(k-1)$-ple point at each k-ple point of the n-ic and passing through $n-2$ other fixed points of the n-ic satisfies the conditions imposed on the N-ic. In fact, if $N = n-2$ and $D = 1$,

$$\begin{aligned} \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\Sigma k(k-1) &= 1, \\ n-2+\tfrac{1}{2}\Sigma k(k-1) &= \tfrac{1}{2}N(N+3)-1, \\ n-2+\Sigma k(k-1) &= nN-2. \end{aligned}$$

In general we may take for the N-ic an adjoined $(n-2)$-ic passing through $n-2$ other fixed points of the n-ic.*

In practice, however, we may often take a lower value of N. For instance, if the n-ic has only ordinary double points, we may take for the N-ic an $(n-3)$-ic through all the double points but one.

Ex. 1. Express the coordinates of any point on a cubic rationally in terms of elliptic functions.

[Consider the intersections of the cubic with a line through a fixed point of the curve.]

Ex. 2. Express rationally in terms of elliptic functions the coordinates of any point on

(i) $xz^2 = y(y-x)(y-k^2x)$.

(ii) $(a^2+b^2)(y-x) = (y+x)(a^2x^2+b^2y^2)$.

(iii) $y^2z = 4x^3-g_2xz^2-g_3z^3$.

(iv) $x^3+y^3+z^3+6mxyz = 0$.

(v) $(x+y+z)^3+6kxyz = 0$.

(vi) $ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2) = 0$.

[Consider the intersections with

$$x = ty,\quad x = ty,\quad x = tz,\quad z = t(x+y),\quad z = t(x+y),\quad y = tx.]$$

Ex. 3. Express the coordinates of any point on a quartic of unit deficiency rationally in terms of elliptic functions.

[Consider its intersections with a line through a node; and so for any n-ic with an $(n-2)$-ple point.]

Ex. 4. Express rationally in terms of elliptic functions the coordinates of any point on the quartics of Ch. XVIII, § 5 (i), § 7 (i), § 14 (ii), § 15, Ex. 1.

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* This follows from the second theorem of Ch. IX, § 10. But we may also argue as follows: Since the coordinates of any point of a curve with unit deficiency having only ordinary multiple points can be expressed rationally in terms of elliptic functions, and any curve with unit deficiency can be transformed into such a curve by rational change of variables (successive quadratic transformations), the coordinates of a point on any curve with unit deficiency can be expressed rationally in terms of elliptic functions.

Ex. 5. Express rationally in terms of elliptic functions the coordinates of any point on

(i) $z^3(x^3+y^3) = x^3y^3$.

(ii) $(u-\alpha y^2)(u-\beta y^2)(u-\gamma y^2) = k(xu-y^3)^2$, where $u \equiv yz+x^2$.

(iii) $z^2u_3+zxyu_2+x^2y^2u_1 = 0$, where u_1, u_2, u_3 are of degrees 1, 2, 3 in x and y.

[Consider the intersections with

$$\text{(i) } yz = tx(y-z), \qquad \text{(ii) } u = ty^2, \qquad \text{(iii) } y = tx.]$$

Ex. 6. The coordinates of any point of a curve are expressed rationally in terms of Weierstrass's function. If $g_2^3 < 27g_3^2$, the curve has a single circuit given by real values of the parameter u. If $g_2^3 > 27g_3^2$, the curve has a second circuit given by values of u for which $u-\omega'$ is real, where $2\omega'$ is an unreal period of Weierstrass's function.

Ex. 7. When the coordinates of a point on a curve with unit deficiency are expressed rationally in terms of Weierstrass's function of a parameter u, we may suppose the sum of the parameters of the intersections of the curve with any other curve to be zero.

[As in § 5, using Abel's theorem on the roots of an equation rational in $\wp u$ and $\wp' u$.]

Ex. 8. The coordinates of any point on an n-ic of deficiency 2 can be expressed rationally in terms of a parameter t and $X^{\frac{1}{2}}$, where X is a polynomial in t of degree 5 or 6.

It is in general impossible to express the coordinates of a point on a curve of deficiency greater than 2 in terms of t and $X^{\frac{1}{2}}$, where X is a polynomial in t.

[(i) Consider the intersection of the n-ic with a $(n-3)$-ic through the nodes.

(ii) As in § 4, Ex. 7. But there are exceptions; e.g. an n-ic with a $(n-2)$-ple point.]

§ 8.

We now prove the converse theorem:

If the coordinates of any point on a curve can be expressed rationally in terms of $\wp u$ *and* $\wp' u$, *the curve is of unit deficiency.*

If (x, y, z) are homogeneous coordinates of any point on the curve, we have

$$x = f(u), \quad y = \phi(u), \quad z = \psi(u);$$

where f, ϕ, ψ are each of the form $A + B\wp' u$, A and B being polynomials in $\wp u$.

Now differentiating repeatedly the relation

$$(\wp' u)^2 = 4(\wp u)^3 - g_2 \wp u - g_3$$

we express $\wp u$, $\wp^2 u$, $\wp^3 u$, ..., $\wp' u \,.\, \wp u$, $\wp' u \,.\, \wp^2 u$, ... linearly in terms of $\wp u$, $\wp' u$, $\wp'' u$, ... For instance

$$\wp^2 u = \tfrac{1}{12}(2\wp'' u - g_2), \quad \wp' u \,.\, \wp u = \tfrac{1}{12}\wp''' u, \text{ \&c.}$$

Hence we may take

$$f(u) = a + a_0 \wp u + a_1 \wp' u + \ldots + a_{n-2} \wp^{(n-2)} u,$$

and so for $\phi(u)$ and $\psi(u)$.

The curve meets $\lambda x + \mu y + \nu z = 0$ where

$$\lambda f(u) + \mu \phi(u) + \nu \psi(u) = 0.$$

The left-hand side of this equation has n poles, and has therefore n zeros.* Hence the curve is of degree n.

The points of contact of any tangent through the point (x, y, z) is given by

$$\lambda x + \mu y + \nu z = 0,$$

where $\lambda = \phi' \psi - \phi \psi', \quad \mu = \psi' f - \psi f', \quad \nu = f' \phi - f \phi'$, (see § 2).

In $\phi' \psi - \phi \psi'$ the terms in $\wp^{(n-2)} u$ $\wp^{(n-1)} u$ cancel, so that λ when reduced to linear form becomes of the type

$$A + A_0 \wp u + A_1 \wp' u + \ldots + A_{2n-2} \wp^{(2n-2)} u,$$

and so for μ, ν.

Hence the class of the curve is $2n$ at most.

The argument of § 4 (p. 148) will show that the deficiency is zero or unity. But f, ϕ, ψ will not usually be rational functions of a single parameter. Hence in general $D = 1$.

Ex. Show by means of § 8 that any point on a cubic may be taken as $(\wp u, \wp' u, 1)$ by a suitable choice of triangle of reference.

[We proved that with any choice of triangle of reference x, y, z are linear functions of $\wp u$ and $\wp' u$. Hence three linear functions of x, y, z can be chosen in the ratios $\wp u : \wp' u : 1$.]

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* Forsyth, *Theory of Functions*, § 116.

CHAPTER XI

DERIVED CURVES

§ 1. Derived Curves.

FROM a given curve may be derived other curves by various geometrical processes, for instance its polar reciprocals, inverses, evolute, pedals, orthoptic locus, &c.

The case of the polar reciprocals has been already sufficiently discussed. In this chapter we shall consider some of the other derived curves. In particular we shall concern ourselves with determining the *type* of the derived curve (as defined by its Plücker's numbers), when the type of the original curve is given. We shall also determine the multiple points and any other peculiarities of the derived curves which may be of interest.

In determining the Plücker's numbers of any derived curve we shall make use of the principle that, if the derived curve is algebraic, the number of its intersections with every line is the same.

Hence, in order to determine its degree, it is sufficient to find the number of its intersections with a single line.

Similarly to find its class, it is sufficient to find the number of its tangents passing through a single point.

§ 2. Evolutes.

The evolute of a curve is the envelope of its normals, the locus of the intersection of each normal with the consecutive normal, and the locus of the centre of curvature at each point of the curve.

Since the centre of curvature at any point of the curve $f(x, y) = 0$ is (ξ, η), where

$$\xi = x + f_1(f_1^2 + f_2^2)/(2f_1f_2f_{12} - f_2^2f_{11} - f_1^2f_{22}),$$
$$\eta = y + f_2(f_1^2 + f_2^2)/(2f_1f_2f_{12} - f_2^2f_{11} - f_1^2f_{22}) \quad . \quad . \quad . \quad \text{(i)}$$

suffixes 1 and 2 denoting partial differentiation with respect to x and y, the evolute of an algebraic curve is algebraic.*

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* Its equation is obtained by eliminating x and y from the three equations just written, and then replacing ξ and η by x and y.

We denote the Plücker's numbers of the curve by n, m, δ, κ, τ, ι, D, and of its evolute by n', m', δ', κ', τ', ι', D'.

We now determine n', m', ... in terms of n, m, ...

Suppose that ω, ω' are the circular points at infinity and that the tangent at P to the given curve meets $\omega\omega'$ in Q, while $(\omega\omega', QR)$ is harmonic. Then PR is the normal at P, i. e. touches the evolute.

If P is on $\omega\omega'$ it coincides with Q, and QR coincides with $\omega\omega'$, which is therefore a tangent to the evolute.

In this case let Q' be a point on $\omega\omega'$ near P, and let $Q'P_1$, $Q'P_2$ be the tangents from Q' to the curve which touch at P_1 and P_2 adjacent to P (Fig. 1).

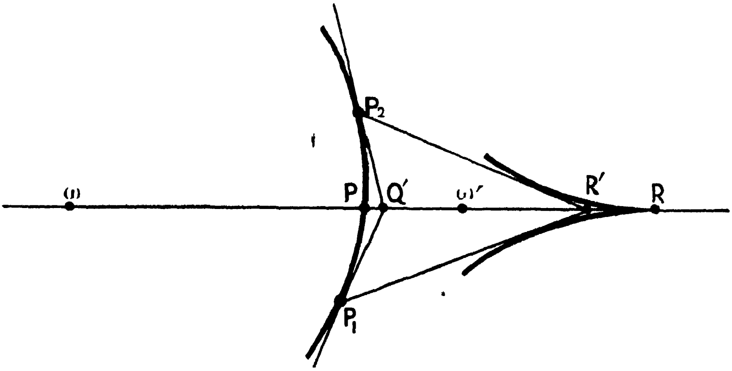

Fig. 1.

Let $(\omega\omega', Q'R')$ be harmonic. Then P_1R' and P_2R' are the normals at P_1 and P_2, so that through R' three consecutive tangents $R'P_1$, $R'P$, $R'P_2$ can be drawn to the evolute, all ultimately coinciding with $\omega\omega'$. Hence R is a cusp of the evolute, at which $\omega\omega'$ is a tangent, and $\omega\omega'$ meets the evolute in three points at R.

We see then that the evolute has n cusps at infinity at which $\omega\omega'$ is the tangent, one for each of the n intersections of $\omega\omega'$ with the given curve, and meets $\omega\omega'$ $3n$ times at these cusps.

The only other infinite points of the evolute are those due to the ι inflexions of the given curve. Hence

$$n' = 3n + \iota.$$

The tangents to the evolute from a point Q on $\omega\omega'$ are the m normals corresponding to the m tangents to the given curve from R, where $(\omega\omega', QR)$ is harmonic, and the line $\omega\omega'$ which counts for n tangents, since it has n points of contact, namely the cusps of the evolute on $\omega\omega'$. Hence

$$m' = m + n.$$

Again, it is evident that in general no two consecutive normals coincide; hence the evolute has not coincident tangents at consecutive points, i. e. has no inflexional tangent. Therefore

$$\iota' = 0.$$

Plücker's equations now give δ', κ', τ'.

Ex. 1. Find the evolute of $y^2 = 4ax$.

[The normal at $(at^2, 2at)$ is $tx + y = a(t^3 + 2t)$, meeting the consecutive normal at $x = a(3t^2 + 2)$. These two equations give

$$x = a(3t^2+2), \qquad y = -2at^3$$

as the centre of curvature at $(at^2, 2at)$. Hence the evolute is

$$27ay^2 = 4(x-2a)^3.]$$

Ex. 2. Find the evolute of

(i) $x = a\cos\phi, \qquad y = b\sin\phi.$

(ii) $x = ct, \qquad y = c/t.$

(iii) $x = at^2, \qquad y = at^3.$

(iv) $x = a\cos^3\phi, \qquad y = a\sin^3\phi.$

[The centre of curvature is

(i) $x = (a^2-b^2)\cos^3\phi/a, \qquad y = (b^2-a^2)\sin^3\phi/b.$

(ii) $x = c(3t^4+1)/2t^3, \qquad y = c(t^4+3)/2t.$

(iii) $x = -\frac{3}{2}at^4, \qquad y = \dfrac{a}{3}(2t+6t^3).$

(iv) $x = a(\cos^3\phi + 3\cos\phi\sin^2\phi), \qquad y = a(3\cos^2\phi\sin\phi - \sin^3\phi).$]

Ex. 3. Find the Plücker's numbers of the evolute of $a^q y^p = x^{p+q}$, p and q being positive integers prime to one another.

[$m' = p + 2q$, $D' = 0$; $n' = p + 2q$ if $p \geqslant q$, $n' = 3q$ if $p \leqslant q$.

For other examples take the curves of Ex. 2.]

Ex. 4. The number of normals which can be drawn from any point to a curve is $m + n$.

[The number is m'. Verify the result by consideration of the fact that the normals from O to a curve are lines joining O to the intersections of the curve with the curve obtained by rotating it through a very small angle about O. See also Ch. XII, § 5, Ex. 12.]

Ex. 5. How many circles of curvature of a given curve have four-point contact?

[Each such circle is given by a cusp of the evolute not lying on $\omega\omega'$ Hence the number is

$$\kappa' - n = 3(n' - m') + \iota' - n = 5n - 3m + 3\iota.]$$

Ex. 6. How many lines are normal to a curve at two points?

[$\tau' - \frac{1}{2}n(n-1) = \frac{1}{2}(m^2 + 2mn - 4m - \kappa)$.]

Ex. 7. How many pairs of concentric circles exist, each of which osculates a given curve?

Ex. 8. The deficiency of a curve and of its evolute are equal.

Ex. 9. The evolute has the same foci and directrices as its involute.

[A tangent passing through ω or ω' coincides with its normal. The curve and evolute touch at the point of contact of such a tangent.]

Ex. 10. There are $m(m+n-4)$ normals to a curve which are also tangents.

[The common tangents of curve and evolute which do not pass through ω or ω'. See Ex. 9.]

Ex. 11. Find Plücker's numbers for the locus of the extremity of the polar subtangent of a given curve, O being the pole.

[The locus is the polar reciprocal of the evolute of the reciprocal with respect to O. Consider the case of a conic with O as focus.]

Ex. 12. Find the Plücker's numbers of the locus of the harmonic conjugate of a variable point P on a given curve with respect to the intersections of the tangent at P with two fixed lines.

[Reciprocate and project the reciprocals of the fixed lines into ω and ω'.]

Ex. 13. If a curve touches $\omega\omega'$ at P, the evolute has an inflexion at R, where $(\omega\omega', PR)$ is harmonic, $\omega\omega'$ being the tangent at the inflexion.

Ex. 14. If a curve passes through ω (or ω'), the evolute has the tangent at ω as inflexional tangent.

Ex. 15. If a curve touches $\omega\omega'$ in g points and passes f times through each circular point,

$$n' = 3n + \iota - 3(2f+g), \qquad m' = m+n-(2f+g), \qquad \iota' = 2f+g.$$

What modification must be made in the results of Ex. 4 to 7 in this case?

[Use Ex. 13 and 14.]

Ex. 16. If the curve has $\omega\omega'$ as inflexional tangent, the evolute has $\omega\omega'$ as a tangent at a point of undulation, and so on.

Ex. 17. The number of normals common to two curves of degree n, n_1 and class m, m_1 is $mm_1 + mn_1 + m_1 n$.

[The normals are the finite common tangents of their evolutes. Putting $n_1 = 0$, $m_1 = 1$ we have the result of Ex. 4.]

§ 3. Inverse Curve.

The curve inverse to $f(x, y) = 0$ with respect to the circle whose centre is the origin and radius k is

$$f\left(k^2 x/(x^2+y^2),\ k^2 y/(x^2+y^2)\right) = 0.$$

If $f \equiv u_0 + u_1 + u_2 + \ldots + u_n$, where u_r is homogeneous of degree r in x and y, the inverse curve is

$$u_0(x^2+y^2)^n + k^2 u_1 (x^2+y^2)^{n-1} + \ldots + k^{2n} u_n = 0.$$

It is at once verified that this inverse curve has a multiple point of order n at each of the circular points ω, ω'* and has also an n-ple point at the origin O.

(See also Ch. IX, §§ 1, 2.)

To find the Plücker's numbers n', m', δ', κ', τ', ι', D' of the curve inverse to a curve with Plücker's numbers n, m, δ, κ, τ, ι, D, we proceed as follows.

First of all, it is obvious from the above that $n' = 2n$.

Secondly, a cusp of the given curve inverts into a cusp of the inverse curve, and vice versa, so that $\kappa' = \kappa$.

Again, a node of the given curve (not at O or on $\omega\omega'$) inverts into a node of the inverse curve. Also the n-ple points of the inverse curve at O, ω, and ω' are each equivalent to $\frac{1}{2}n(n-1)$ nodes, so far as their effect on Plücker's numbers are concerned (Ch. VIII, § 3). Hence

$$\delta' = \delta + \tfrac{3}{2}n(n-1).$$

We can now deduce m', τ', ι', D'. We find

$$\left.\begin{array}{llll} n' = 2n, & m' = m+2n, & \delta' = \frac{3}{2}n(n-1)+\delta, & \kappa' = \kappa, \\ \tau' = 2n(2n-7)+4mn+2\tau, & & \iota' = 3n+\iota, & D' = D \end{array}\right\}.$$

As a verification we shall determine m' independently.

The tangents from O to the inverse curve are the n tangents at O each counted twice (Ch. VII, § 5) and the tangents from O to the original curve. Hence $m' = m+2n$.

The same result follows from consideration of the tangents from ω (or ω'), remembering that a line through ω inverts into a line through ω'.

The result $D' = D$ is a particular case of the theorem that two curves with a 1 : 1 correspondence have the same deficiency (Ch. XXI, § 3).

We have proved earlier (Ch. V, § 4) that the inverses of the foci of a curve are the foci of the inverse curve; and that, if O is a focus of the curve, the inverse curve has cusps at ω and ω'.

Ex. 1. If the original curve has a k-ple point at O and a p-ple point at each of ω and ω', we have

$$n' = 2n-2p-k, \quad \delta' = \tfrac{1}{2}(n-2p)(n-2p-1)+(n-p-k)(n-p-k-1)+\delta,$$
$$\kappa' = \kappa;$$

and the inverse curve has a $(n-2p)$-ple point at O and $(n-p-k)$-ple points at each of ω and ω'.

Ex. 2. What modification is required in the result of § 3, if the curve touches $\omega\omega'$?

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* In fact any line $x \pm iy = c$ meets the inverse curve at n finite points only; whereas the curve is of degree $2n$.

Ex. 3. A curve of degree $2N$ with N-ple points at ω and ω' inverts into a curve of the same type.

Ex. 4. Find the number of circles of curvature of a given curve which pass through a given point O. Discuss the case in which O lies on the curve.

[Invert with respect to O.]

Ex. 5. Find the number of circles passing through a given point and having double contact with a given curve.

Ex. 6. Find the number of circles passing through two given points and touching a given curve.

[Invert with respect to either point.]

Ex. 7. An n-ic is self-inverse with respect to a circle j. Show that n is even, and that the curve has a $\frac{1}{2}n$-ple point at each circular point. Show also that the curve has $\frac{1}{2}n(n-2)$ foci lying on j in general, which are the intersections of j with the locus of the centres of a family of circles having double contact with the n-ic.

Ex. 8. A $2n$-ic has n-ple points at ω and ω'. Any transversal through P meets it in $Q_1, Q_2, \ldots, Q_{2n}$. Show that the product

$$p_P \equiv PQ_1 \,.\, PQ_2 \,.\, \ldots\, PQ_{2n}$$

is independent of the direction of the transversal; and that on inversion with respect to O we have

$$p'_{P'} = k^{4n} \,.\, p_P / p_O \,.\, OP^{2n}.$$

[(i) Use polar coordinates. (ii) Take OP as the transversal.]

§ 4. Pedal Curve.

If OY is the perpendicular from a fixed point O on the tangent at any point P of a given curve, the locus of Y is the *pedal* (*first pedal*) of the given curve with respect to O. The pedal of the pedal is called the *second pedal*, the pedal of the second pedal is called the *third pedal*, and so on.

The angles between the radius vector and the tangent at corresponding points of a curve and its pedal are equal.

For let the tangents at consecutive points P, P' of the curve meet at T, and let OY, OY' be the perpendiculars from O on these tangents (Fig. 2).

Then since O, Y', Y, T are concyclic, the angles OTY', OYY' are equal.

But in the limit these angles are the angle OPY and the angle between OY and the tangent to the pedal at Y.

The pedal is the envelope of the circles described on the radii vectores as diameters.

For by the last theorem the circle OPY touches the pedal at Y.

We now proceed to find the Plücker's numbers n', m', δ', κ',

τ', ι', D' of the pedal, the Plücker's numbers n, m, δ, κ, τ, ι, D of the curve being supposed given.

The pedal is the inverse of the polar reciprocal of the curve with respect to a circle whose centre is O. For if Q is the pole of PY in Fig. 2 with respect to this circle, OYQ is a straight line and $OY \cdot OQ$ is constant. We have then only to interchange m and n, δ and τ, κ and ι in the expressions for n', m', ... obtained in § 3. We have

$$\left.\begin{array}{llll} n' = 2m, & m' = n+2m, & \delta' = \frac{3}{2}m(m-1)+\tau, & \kappa' = \iota, \\ \tau' = 2m(2m-7)+4mn+2\delta, & & \iota' = 3m+\kappa, & D' = D \end{array}\right\}.$$

Also the pedal has multiple points of order m at O, ω, and ω'.

As a verification, we note that it is evident geometrically that there is a branch of the pedal through O corresponding

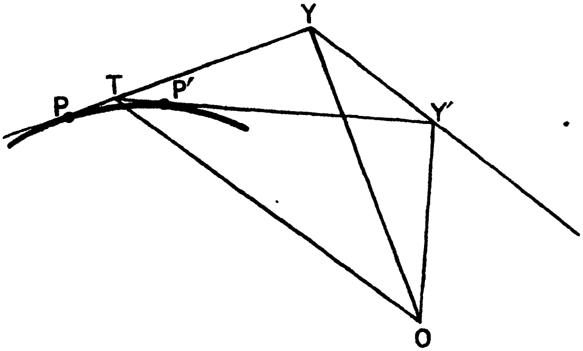

Fig. 2.

to each tangent from O to the given curve. Hence O is an m-ple point of the pedal.

Again, any line through O meets the pedal m times at O and m times where it intersects the m perpendicular tangents of the given curve, so that $n' = 2m$.

Also the tangents from O to the pedal are the m tangents at O each counted twice and the perpendiculars from O to the n asymptotes of the given curve, as will be evident from the first theorem of this section.

Hence $m' = n+2m$.

The same theorem will show that to each bitangent of the given curve corresponds a node of the pedal and to each inflexional tangent corresponds a cusp; so that $\kappa' = \iota$.

The reader will readily prove that to each of the m tangents from ω to the given curve corresponds a tangent at ω to the pedal, this tangent at ω and $\omega\omega'$ being harmonic conjugates with respect to $O\omega$ and the tangent from ω to the given curve.* Hence ω and ω' are m-ple points of the pedal; so that $\delta' = \frac{3}{2}m(m-1)+\tau$.

Ex. 1. If the polar equation of the pedal of a curve is $r = f(\theta)$, the polar equation of the pedal with respect to (c, α) as pole is

$$r + c\cos(\theta - \alpha) = f(\theta).$$

Ex. 2. The pedal of a parabola with respect to the vertex is a cuspidal cubic.

[The pedal with respect to the focus is the tangent at the vertex: now use Ex. 1.

Otherwise: Any tangent to $y^2 = 4ax$ is $x - ty + at^2 = 0$, and the perpendicular from the origin is $tx + y = 0$. Now eliminate t.]

Ex. 3. The pedal of a circle is a limaçon.

[Use Ex. 1.]

Ex. 4. The pedal of $r^m = a^m \cos m\theta$ is obtained by changing m into $m/(m+1)$.

Ex. 5. The pedal of a conic with respect to its centre is a unicursal quartic.

Ex. 6. The line joining O to a focus of a curve is bisected by a singular focus of the pedal.

The ordinary foci of the pedal are the feet of the perpendiculars from O on the lines joining the intersections of the curve with the circular lines through O.

Ex. 7. Any tangent to the curve of Ex. 1 is

$$x\cos\omega + y\sin\omega = f(\omega).$$

The normal is

$$-x\sin\omega + y\cos\omega = f'(\omega),$$

and the corresponding normal to the evolute is

$$x\cos\omega + y\sin\omega = -f''(\omega).$$

The coordinates of the point of contact of the tangent and of the centre of curvature are

$$(\cos\omega \,.\, f - \sin\omega \,.\, f', \quad \sin\omega \,.\, f + \cos\omega \,.\, f')$$

and

$$(-\sin\omega \,.\, f' - \cos\omega \,.\, f'', \quad \cos\omega \,.\, f' - \sin\omega \,.\, f'').$$

The radius of curvature is $f(\omega) + f''(\omega)$.

Ex. 8. Tangents are drawn in any direction to a given curve. Show that the algebraic sum of the radii of curvature at the points of contact is zero. Show also that the centroid O of the points of contact coincides with the centroid of the corresponding centres of curvature, and is independent of the direction of the tangents.

[If the polar equation of the pedal is

$$r^m + r^{m-1}u_1 + r^{m-2}u_2 + \ldots + u_m = 0,$$

where u_k is homogeneous of degree k in $\cos\theta$ and $\sin\theta$, the sum of the radii of curvature is by Ex. 7 $-\frac{d^2u_1}{d\theta^2} - u_1$, which is zero.]

---

* Draw a diagram in which the tangent to the given curve at P passes close to ω, construct Y, and proceed to the limit.

Ex. 9. The sum of the perpendiculars from the point O of Ex. 8 on the tangents drawn in any direction is zero, and O is the centroid of the foci.

[We have $u_1 \equiv 0$, if we take O as pole.]

Ex. 10. The deficiencies of a curve and its pedal are the same.

Ex. 11. Show that, if a curve touches $\omega\omega'$ at H, its pedal with respect to O has in general an asymptote perpendicular to OH, whose distance from O is the same for all positions of O.

Discuss the Plucker's numbers of the curve.

Ex. 12. Discuss the Plücker's numbers of the pedal of a curve with multiple points at O, ω, and ω'.

Ex. 13. Find the Plucker's numbers of the second, third, ... pedals of a curve.

[See *Messenger of Math.*, July 1904, p. 50.]

Ex. 14. If the curve a is the r-th pedal of the curve b, b is called the r-th *negative pedal* of a.

Find the Plucker's numbers of the first negative pedal of a given curve.

[It is the polar reciprocal of the inverse curve with respect to O.]

Ex. 15. Show that the inverse to the r-th positive pedal is the r-th negative pedal of the inverse curve.

Ex. 16. The locus of the intersection of any tangent to a curve with the line through O making a fixed angle with the tangent is similar to the pedal.

Ex. 17. Find the Plücker's numbers of the locus of the centre of a circle passing through a given point O and touching a given curve.

[Invert with respect to O.]

Ex. 18. Find the Plücker's numbers of the envelope of a circle which passes through a given point O and whose centre lies on a given curve.

[The envelope and the pedal with respect to O have O as centre of similitude.]

§ 5. Orthoptic Locus.

The locus of the intersection of two perpendicular tangents of a curve is called its *orthoptic locus*.

If the coordinates used are rectangular Cartesian, we may obtain the equation of the orthoptic locus as follows. Let $f(\lambda, \mu) = 0$ be the tangential equation of the curve, and let $\lambda x + \mu y + 1 = 0$ be the tangential equation of a point R on the orthoptic locus, so that two of the tangents from R to the curve are perpendicular. If we make $f(\lambda, \mu) = 0$ homogeneous in λ, μ by means of $\lambda x + \mu y + 1 = 0$, the resulting equation in $-\lambda/\mu$ gives the slope of the tangents from R to the given curve. The product of two of the roots of this equation in $-\lambda/\mu$ must therefore be -1. If we write down (Ch. I, § 11) the condition that this should be the case, we obtain a relation between x and y which is the point-equation of the orthoptic locus.

We see at once that the orthoptic locus of an algebraic curve is algebraic.

If the equation of the curve is given in parametral form, we can express the equations of the tangents at the points with parameters t and T in the form

$$x+f(t)y = \phi(t) \text{ and } x+f(T)y = \phi(T).$$

If these are perpendicular, $f(t) \, . f(T)+1 = 0$; and from the three equations just written we can express the coordinates x, y of any point on the orthoptic locus in terms of a single parameter.*

If RP, RQ are perpendicular tangents to a curve, the circle PRQ touches the orthoptic locus at R.

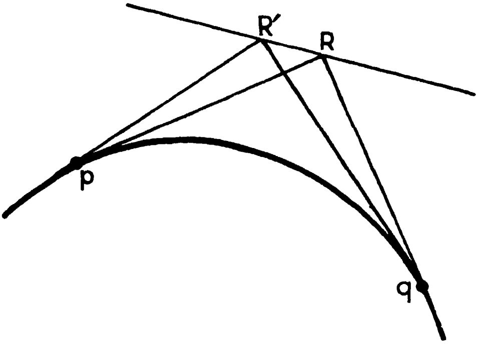

Fig. 3.

For suppose R, R' are consecutive points on the orthoptic locus (Fig. 3), while Rp, Rq and $R'p$, $R'q$ are the perpendicular tangents from R and R'. Then p, q, R, R' are concyclic. If we proceed now to the limit, p, q, R' approach P, Q, R respectively; and the theorem is proved.

Ex. 1. Find the orthoptic locus of the following curves:

(i) Parabola.
(ii) Central conic.
(iii) Circle.
(iv) $y^2 = x^3$.
(v) $3(x+y) = x^3$.
(vi) $x^2y^2-4a(x^3+y^3)+18a^2xy-27a^4 = 0$.

---

* It is usually advisable to express t, T in terms of $u = t+T$, $v = tT$. The three equations then give x, y conveniently in terms of u or v.

(vii) $27(x^2+y^2) = (2x+1)^3$.
(viii) $\lambda^4+\mu^4 = \lambda(\lambda-\mu)(2\lambda-\mu)$.
(ix) $\lambda^4+2\mu^4 = \lambda^2\mu$.
(x) $(\lambda^2-2\mu^2)^2+\lambda\mu = 0$.
(xi) $\lambda^4+2\mu^4 = \lambda\mu(\lambda-\mu)$.
(xii) $\lambda^6+\mu^5 = \lambda^2\mu^2$.
(xiii) $\lambda^5+\lambda^4\mu+\mu^5 = \lambda\mu(\lambda^2-\mu^2)$.

[(i) Straight line.
(ii) Concentric circle.
(iii) Concentric circle.
(iv) $729y^2 = 108x-16$.
(v) $81y^2(x^2+y^2)-36(x^2-2xy+5y^2)+128 = 0$
(vi) $x+y+2a = 0$.
(vii) $16y^4-16xy^2+4x^2+4y^2-4x-1 = 0$.
(viii) $6x(x-y)(2x-y)+17x^2-18xy+9y^2 = 0$.
(ix) $2x^2+y+3 = 0$.
(x) $(x^2+y^2)(4x^2+y^2)+9(2xy+1) = 0$.
(xi) $2x^2-3xy+y^2-x+y+3 = 0$.
(xii) $(x-y)^2+2(x+y)+4 = 0$.
(xiii) $(x-y-1)(2x-y-1)+1 = 0$.]

Ex. 2. Find the directrix of the parabola

$$x = at^2+2bt+c, \qquad y = At^2+2Bt+C.$$

Ex. 3. Show how to find the orthoptic locus when the polar equation $r = f(\theta)$ of a pedal is given.

[Two perpendicular tangents may be written

$$x\cos\alpha+y\sin\alpha = f(\alpha),$$
$$-x\sin\alpha+y\cos\alpha = (-1)^k f\{(2k+1)\frac{\pi}{2}+\alpha\};$$

from which we get x, y in terms of α.]

Ex. 4. Find the orthoptic locus of the curve $r^m = a^m\cos m\theta$.

[$f(\theta) = a\cos^n\theta/n$, where $n = (m+1)/m$.]

Ex. 5. Find the orthoptic locus of the hypocycloid.

[$f(\theta) = c\sin m\left(\frac{\pi}{2}-\theta\right)$, where $c = a-2b$ and $m = a/(a-2b)$; a being the radius of the fixed, and b the radius of the rolling circle.]

Ex. 6. The orthoptic locus of a cardioid is a circle and a limaçon.

[In Ex. 5 $a+b = 0$.]

§ 6. Plücker's Numbers of Orthoptic Locus.

We proceed now to determine the Plücker's numbers n', m', δ', κ', τ', ι', D' of the orthoptic locus of a given curve with Plücker's numbers n, m, δ, κ, τ, ι, D.

If two points E and F divide harmonically the line joining the circular points ω and ω', a tangent from E and a tangent from F to the given curve are perpendicular and meet at R on the orthoptic locus. Suppose now E, and therefore F, approaches ω. Then R also approaches ω, while $R(\omega\omega', EF)$ remains harmonic; unless the tangents from E and F become consecutive, when R becomes the point of contact of either tangent.

Proceeding to the limit we see that to each pair of tangents to the given curve from ω corresponds a branch of the orthoptic locus through ω, while the tangent to this branch and the line $\omega\omega'$ are harmonic conjugates with respect to the pair of tangents in question. Now there are $\frac{1}{2}m(m-1)$ such pairs of tangents; so that the orthoptic locus has a $\frac{1}{2}m(m-1)$-ple point at each of ω and ω'. It is readily seen that there is no other point of the orthoptic locus on $\omega\omega'$ in general, so that $n' = m(m-1)$.

Again, the theorem of § 5 shows that the intersections of a bitangent of the given curve with the m perpendicular tangents are nodes of the orthoptic locus, while the intersections of an inflexional tangent with the m perpendicular tangents are cusps of the locus. In general there are no other cusps, so that $\kappa' = m\iota$.

We now proceed to the more difficult task of finding m'. It suffices to find the number of tangents which can be drawn to the locus from ω'.

Of these tangents $m(m-1)$ coincide with the tangents at ω'. To find the remaining tangents from ω', suppose that in Fig. 3 RR' passes through ω'. Then since $R(pq, \omega\omega')$ and $R'(pq, \omega\omega')$ are harmonic, pq passes through ω.

Proceeding to the limit, p, q become the points of contact P, Q of the perpendicular tangents RP, RQ; while the tangent at R to the orthoptic locus passes through ω' and PQ through ω.

Now suppose any line whatever through ω meets the given curve in P and Q. Let the tangents at P and Q meet in T; and consider the envelope of the line TV where $T(PQ, \omega V)$ is harmonic. By what has just been said, every tangent from ω' to the envelope which does not coincide with $\omega\omega'$ will be a tangent to the orthoptic locus. Now in general the envelope will not touch $\omega\omega'$; and hence the number of tangents from ω' to the orthoptic locus, whose points of contact are not on $\omega\omega'$, is equal to the class of the envelope.

To find this class, we obtain the number of tangents from ω to the envelope. The line TV cannot pass through ω unless ωPQ touches the given curve at P but not at Q, or vice versa. If this is the case, however, ωPQ will touch the envelope at a point U such that $(\omega U, PQ)$ is harmonic. Hence each tangent from ω to the curve is an $(n-2)$-ple tangent to the envelope; so that the class of the envelope is $m(n-2)$.

We have then

$$m' = m(n-2) + m(m-1) = m(m+n-3).$$

We now obtain δ', τ', ι', D' from Ch. VIII, § 1 and see that

$$\left.\begin{aligned} n' &= m(m-1), \quad m' = m(m+n-3), \\ \delta' &= \tfrac{1}{2}m\{(m+1)(m-2)^2+2\tau\}, \quad \kappa' = m\iota, \\ \tau' &= \tfrac{1}{2}m\{m(m+n)^2-(6m^2+6mn+n^2)-m+22+2\delta\}, \\ \iota' &= m(3m+\kappa-6), \quad D' = \tfrac{1}{2}(m-1)(m-2)+mD \end{aligned}\right\}.$$

In general the orthoptic locus of a curve is of relatively high degree. For instance, the general cubic has an orthoptic locus of degree 30. But as will be seen in the following examples, the orthoptic locus simplifies materially, if the given curve is specialized by touching the line $\omega\omega'$, &c.

Ex. 1. The singular foci of the orthoptic locus are the $\frac{1}{2}m(m-1)$ middle points of the $\frac{1}{2}m(m-1)$ segments joining the m real foci of the curve.

Ex. 2. There are $m(m-3)(n-2)$ points of a curve from which two perpendicular tangents can be drawn to the curve, neither of which coincides with the tangent at the point.

[The curve meets the orthoptic locus at the points of contact of the tangents to the curve from ω and ω, and touches it at the feet of the $m(m+n-4)$ normals which are also tangents. There remain

$$m(m-3)(n-2)$$

intersections of curve and locus.]

Ex. 3. There are $\frac{1}{4}m(m+1)(m-2)(m-3)$ points from which two pairs of mutually perpendicular tangents can be drawn to a given curve of class m.

[The nodes of the orthoptic locus other than ω, ω', and the $m\tau$ nodes derived from bitangents of the given curve.]

Ex. 4. If a curve touches $\omega\omega'$ at Y and $(YZ, \omega\omega')$ is harmonic, the orthoptic locus has ω and ω' as $\frac{1}{2}(m-1)(m-2)$-ple points and Z as an $(m-1)$-ple point. If Y is a cusp at which $\omega\omega'$ is a tangent, the tangents at Z to the locus are the tangents from Z to the curve other than $\omega\omega'$.

Ex. 5. If $\omega\omega'$ is a k-ple tangent to the curve,

$$n' = (m-k)(m-1), \qquad m' = (m-k)(m+n-3-k), \qquad \kappa' = (m-k)\iota.$$

Ex. 6. If in Ex. 4 the curve has $\omega\omega'$ as inflexional tangent at Y, the locus has $(m-2)$ linear branches touching $\omega\omega'$ at Z. We have

$$n' = (m-1)(m-2), \qquad m' = (m-2)(m+n-4), \qquad \kappa' = (m-2)(\iota-1).$$

Ex. 7. If the curve touches $\omega\omega'$ at Y and Z where $(\omega\omega', YZ)$ is harmonic, the locus has Y and Z as $(m-3)$-ple points, ω and ω' as $\frac{1}{2}(m-2)(m-3)$-ple points. It also cuts $\omega\omega'$ at one other point.

Discuss the cases in which Y, Z are cusps, inflexions, &c.

Ex. 8. If the curve passes through ω and ω', the orthoptic locus has a branch through ω touching the curve at ω, and $m-2$ superlinear branches of order 2 at ω. We have

$$n' = m(m-1), \qquad m' = m(m+n-5)+4, \qquad \kappa' = 2(m-2)+m\iota.$$

Ex. 9. If bitangents of the curve pass through ω and ω',

$$n' = (m+1)(m-2), \qquad m' = m(m+n-3)-4, \qquad \kappa' = m\iota.$$

Ex. 10. If inflexional tangents of the curve pass through ω and ω',

$$n' = (m+1)(m-2), \qquad m' = m(m+n-4), \qquad \kappa' = m\iota-4.$$

Ex. 11. The degree and class of the locus of the intersection of two perpendicular normals are

$$(m-1)(m+n-2) \quad \text{and} \quad (m-1)(4m+\kappa-6).$$

The locus has no cusp.

[The locus is the orthoptic locus of the evolute.]

Ex. 12. Verify the results of Ex. 1 to 11 on the curves of § 5, Ex. 1.

Ex. 13. The orthoptic locus is a straight line when the curve (i) is a parabola; (ii) touches $\omega\omega'$ in two points dividing $\omega\omega'$ harmonically, while $n = 4$, $m = 3$.

Ex. 14. The orthoptic locus is a circle when the curve (i) is a circle, (ii) is a central conic, (iii) touches $\omega\omega'$ at ω and ω', while $n = 4$, $m = 3$.

Ex. 15. The orthoptic locus is a parabola when the curve (i) has $\omega\omega'$ as inflexional tangent, while $n = m = 3$; (ii) has $\omega\omega'$ as ordinary and inflexional tangent, the points of contact dividing $\omega\omega'$ harmonically, while $n = 5$, $m = 4$; (iii) has $\omega\omega'$ as inflexional tangent at two points dividing $\omega\omega'$ harmonically, while $n = 6$, $m = 5$.

Ex. 16. The orthoptic locus is a central conic when the curve (i) has $\omega\omega'$ as bitangent, while $n = 4$, $m = 3$; (ii) has $\omega\omega'$ as triple tangent, two of the points of contact dividing $\omega\omega'$ harmonically, while $n = 6$, $m = 4$; (iii) has $\omega\omega'$ as quadruple tangent, two pairs of points of contact dividing $\omega\omega'$ harmonically, while $n = 8$, $m = 5$.*

§ 7. Isoptic Locus.

The locus of the intersection of two tangents to a curve which are inclined at a fixed angle α is called an *isoptic locus*. The investigation of its properties is similar to that of §§ 5 and 6. The construction for the tangent to the isoptic locus implied in Fig. 3 still holds. The locus is an algebraic curve if we consider tangents cutting at an angle $\pi-\alpha$ as included among those which cut at an angle α.†

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* The orthoptic locus is a cubic in fourteen cases, and a quartic in thirty-eight cases.

† For the circle the loci of the intersection of two tangents cutting at angles α or $\pi-\alpha$ are distinct and both algebraic (being in fact concentric circles). But this is not usually the case. For the parabola the two loci are the two branches of the same hyperbola.

For the Plücker's numbers n', m', ... of an isoptic locus we find

$$\left.\begin{aligned} &n' = 2m(m-1), \quad m' = m(2n+2m-4), \\ &\delta' = m(2m-3)(m^2-m-1)+2m\tau, \quad \kappa' = 2m\iota, \\ &\tau' = m\{2m(m+n)^2-(8m^2+8mn+n^2)-2m+12+2\delta\}, \\ &\iota' = 2m(3m+\kappa-3), \quad D' = (m-1)^2+2mD \end{aligned}\right\}$$

An isoptic locus has $m(m-1)$-ple points at ω and ω'.

Ex. 1. Tangents RP, RQ to a curve are inclined at a constant angle, and the normals at P and Q meet at S. Show that RS is the normal at R to the locus of R.

Deduce the orthoptic locus of a parabola or central conic.

[S is the instantaneous centre of rotation of the rigid body PQR. The line RS is parallel to the axis or passes through the centre. Therefore the orthoptic locus is a straght line or concentric circle.]

Ex. 2. The isoptic locus touches the curve at the $2m$ points of contact of tangents from ω and ω' (cuts if $\alpha = \frac{1}{2}\pi$), and at the $2m(m+n-4)$ points P such that a line through P making an angle α with the tangent at P touches the curve elsewhere. It cuts the curve at the $2m(m-3)(n-2)$ points Q of the curve from which two tangents can be drawn inclined at an angle α, neither touching at Q.

Ex. 3. Find the locus of the intersection of two tangents inclined at a constant angle, one drawn to each of two given curves.

[The isoptic locus of the two curves taken together, less the isoptic loci of the two curves taken separately.]

Ex. 4. Find the isoptic loci of

$$y^2 = 4ax \quad \text{and} \quad x^2/a^2+y^2/b^2 = 1.$$

$$[y^2-4ax = \tan^2\alpha\,(x+a)^2 \quad \text{and}$$
$$\tan^2\alpha\,(x^2+y^2-a^2-b^2)^2 = 4(b^2x^2+a^2y^2-a^2b^2).]$$

Ex. 5. Find the Plücker's numbers of the envelope of a chord of a given curve subtending an angle of given magnitude at a given point O.

[Reciprocate with respect to O.]

Ex. 6. A curve has k-fold symmetry about O. Show that the tangents at the ends of the radii through O inclined at an angle α, which is a multiple of $2\pi/k$, meet on a curve which is part of an isoptic locus and which is similar to the pedal with respect to O.

§ 8. Cissoid.

Suppose that from a fixed point O any line is drawn cutting two fixed curves Σ_1 and Σ_2 respectively in P_1 and P_2. Take a point P on the line such that $OP = OP_1 - OP_2$. The locus of P is called the *cissoid* of Σ_1 and Σ_2 for the pole O.

Suppose the Plücker's numbers of Σ_1 and Σ_2 are n_1, m_1, ... and n_2, m_2, ..., while the Plücker's numbers of the cissoid are n, m, ... On any line through O there are n_1 points such as

P_1 and n_2 such as P_2. Hence the line meets the cissoid at $n_1 n_2$ points other than O.

Moreover, the line joining O to an intersection of Σ_1 and Σ_2 is evidently a tangent at O to the cissoid; so that O is a $n_1 n_2$-ple point of the cissoid. Therefore any line through O meets the cissoid in $2n_1 n_2$ points.

Hence

$$n = 2n_1 n_2.$$

Again, each of the m_1 tangents from O to Σ_1 is evidently a n_2-ple tangent to the cissoid, and so for the m_2 tangents to Σ_2.

Also the $n_1 n_2$ tangents to the cissoid at O count as $2n_1 n_2$ tangents from O. Hence $n_1 m_2 + n_2 m_1 + 2n_1 n_2$ tangents can be drawn from O to the cissoid, or

$$m = n_1 m_2 + n_2 m_1 + 2n_1 n_2.$$

Also it is evident from a diagram that a line joining O to a cusp of Σ_1 passes through n_2 cusps of the cissoid, and so for Σ_2. Hence

$$\kappa = n_1 \kappa_2 + n_2 \kappa_1.$$

It has been assumed that neither curve passes through the circular points, that all the intersections of the curves are finite, &c.

Ex. 1. Prove that the cissoid of algebraic curves is algebraic.

[This was *assumed* in § 8.]

Ex. 2. The cissoid has n_2 linear branches touching Σ_1 at each of its infinite points.

Ex. 3. Find the Plücker's numbers of the locus of P if in § 8

$$OP = k_1 OP_1 + k_2 OP_2,$$

where k_1 and k_2 are constants.

[Consider the cissoid of the two curves obtained by increasing the radii vectores of Σ_1 and Σ_2 in the ratio $1/k_1$ and $-1/k_2$ respectively. Note the case $k_1 = k_2 = \frac{1}{2}$.]

Ex. 4. If a radius vector meets three given curves in P_1, P_2, P_3 and

$$OP = k_1 OP_1 + k_2 OP_2 + k_3 OP_3,$$

where $PP_1P_2P_3$ is a line and k_1, k_2, k_3 are constants, find the Plücker's numbers of the locus of P. Extend to the case of N given curves.

[First apply Ex. 3 to the locus of Q, where

$$OQ = k_1 OP_1 + k_2 OP_2,$$

and then apply Ex. 3 to the loci of Q and P_3.]

Ex. 5. What modification is necessary in the results of § 8, if either of the given curves passes through O, or if the curves meet at infinity?

Ex. 6. Any transversal OPP_1P_2 through O meets a given curve in P_1 and P_2. Find the Plücker's numbers of the locus of P, if

$$OP = k_1 OP_1 + k_2 OP_2.$$

[See Ex. 3.]

Ex 7. How many chords of a given curve through O are divided in a given ratio at O?

Ex. 8. How many chords of a given curve through O are divided in a given ratio by a given line?

Ex. 9. The cissoid of two circles, one of which passes through O, is a bicircular quartic with a node at O.

Ex. 10. If O, A are two points, find the cissoid of the line through A perpendicular to OA and the circle on OA as diameter.

[If O is (0, 0) and A is $(a, 0)$, the curve is $y^2(a-x) = x^3$. The area between the circle and cissoid is something like an ivy leaf. Hence the name 'cissoid', from the Greek κισσός.]

Ex. 11. The normals at P_1, P_2, and P to Σ_1, Σ_2 and the cissoid meet the perpendicular to OP through O in G_1, G_2, and G. Prove that

$$OG = OG_1 - OG_2.$$

[The polar subnormal is $\dfrac{dr}{d\theta}$ in polar coordinates.]

Ex. 12. If the tangents at P_1, P_2 to Σ_1, Σ_2 meet at T and P_1 bisects QT, the tangent at P to the cissoid passes through Q.

[Take OP and OG in Ex. 11 as axes of reference.]

§ 9. Conchoid.

If on the radius vector OP of a curve we measure off PQ equal to $\pm k$, where k is constant, the locus of Q is called a *conchoid* of the curve. It is the cissoid of the curve and the circle with centre O and radius k.

If we are given the Cartesian equation of a curve, we may obtain the equation of any conchoid as follows. Turn the given equation into polars. Replace r by $r+k$. Expand the powers of $r+k$ which occur by the binomial theorem, and collect terms involving an odd power of k on one side of the equation and the terms involving an even power of k on the other. Now square both sides, and turn the equation back into Cartesian coordinates. We have the required equation of the conchoid.

For instance, consider the case of a curve of degree $2n$ and class m with multiple points of order n at O and at the circular points ω and ω'.

Taking $n = 3$ (the method is general) the equation of the curve is

$$(x^2+y^2)^3+3u_1(x^2+y^2)^2+3u_2(x^2+y^2)+u_3=0,$$

or in polar coordinates

$$r^3+3v_1r^2+3v_2r+v_3=0;$$

where v_p is a homogeneous function of degree p in $\cos\theta$ and $\sin\theta$, and $u_p\,(=r^p v_p)$ is the same function of x and y.

The conchoid is

$$(r+k)^3+3v_1(r+k)^2+3v_2(r+k)+v_3=0$$

or

$$\begin{aligned}\{(r^3+3v_1r^2+3v_2r+v_3)+3k^2(r+v_1)\}^2&\\ =k^2\{3(r^2+2v_1r+v_2)+k^2\}^2&.\end{aligned}$$

Turning back into Cartesians we have

$$\begin{aligned}\{[(x^2+y^2)^3+3u_1(x^2+y^2)^2+3u_2(x^2+y^2)+u_3]&\\ +3k^2(x^2+y^2)(x^2+y^2+u_1)\}^2&\\ =k^2(x^2+y^2)\{3[(x^2+y^2)^2+2u_1(x^2+y^2)+u_2]+k^2(x^2+y^2)\}^2&.\end{aligned}$$

We see that the conchoid is of degree $4n$ with $2n$-ple points at O, ω, ω'. On finding the unreal asymptotes in the usual manner we see that the conchoid has n cusps at ω whose tangents are the tangents at ω to the given $2n$-ic, and so for ω'.

The tangents from O to the conchoid are the $m-2n$ tangents from O to the $2n$-ic not touching at O, each reckoned twice (as is obvious from a diagram), and the $2n$ tangents at O to the conchoid each reckoned twice. Hence the class of the conchoid is $2m$.

Each of the κ cusps of the $2n$-ic gives two cusps of conchoid, which with the cusps of the conchoid at ω and ω' make up $2\kappa+2n$ cusps in all.

Since the conchoid is of degree $4n$, is of class $2m$, and has $2\kappa+2n$ cusps, it has

$$\tfrac{1}{2}\{4n(4n-1)-2m-3(2\kappa+2n)\}$$

nodes.

These are accounted for as follows: by Ch. VIII, § 4 O counts for $n(2n-1)$ nodes, while ω counts for $2n(n-1)$ nodes (and n cusps), and so for ω'.

Moreover, each of the $\delta-\frac{3}{2}n(n-1)$ nodes of the $2n$-ic other than O, ω, ω' gives rise to two nodes of the conchoid. There remain $n(n-1)$ nodes of the conchoid which lie at the middle points of the $n(n-1)$ chords of the $2n$-ic which pass through O and are of length $2k$.*

Ex. 1. Find the conchoid of a straight line.

[If the line is $x = a$, the conchoid is

$$(x-a)^2(x^2+y^2) = k^2x^2.$$

It is called the 'conchoid of Nicomedes'. It has a node at O and a tacnode at infinity with the given line as tangent.]

Ex. 2. Find the conchoid of a circle through O.

[A limaçon $r = k + b\cos\theta$.]

Ex. 3. Find the conchoid of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c = 0.$$

$$[\{(x^2+y^2)(ax^2+2hxy+by^2+2gx+2fy+c)+k^2(ax^2+2hxy+by^2)\}^2$$
$$= 4k^2(x^2+y^2)(ax^2+2hxy+by^2+gx+fy)^2.$$

Consider the case $a = b = 1$, $h = 0$.]

Ex. 4. An n-ic does not pass through O, ω, or ω'. Find the nature of its conchoid at O and at infinity.

[The conchoid is a $4n$-ic with a $2n$-ple point at O, an n-ple point at each of ω and ω', and a tacnode at each infinite point of the n-ic. What are the asymptotes?]

Ex. 5. How many lines can be drawn through a given point on which two given curves intercept a segment of given length?

[Consider the intersections of one curve with a conchoid of the other.]

Ex. 6. The centres of curvature at corresponding points of all possible conchoids of a given curve lie on a conic.

[Use Savary's theorem on the centre of curvature of a roulette.]

§ 10. Parallel Curves.

If along the normal at P to a given curve we measure $PQ = \pm k$, the locus of Q is said to be *parallel* to the given curve. All parallel curves have the same normals and the same evolute. The loci of the two possible positions of Q are in general parts of the same algebraic curve.† The tangents at P and Q are parallel and at a distance k apart. Hence the pedal of the parallel curve is the conchoid of the pedal of the given curve for any pole O.

We proceed to find Plücker's numbers for any curve parallel to a curve of given type. Suppose that the given curve does not pass through O, ω, or ω', and that its Plücker's numbers are n, m, δ, κ, τ, ι. We shall denote the Plücker's numbers of any other curve, which we are for the moment considering, by n', m', δ', κ', τ', ι'.

---

* The $2n$-ic meets the conchoid obtained by changing k into $2k$ in $8n^2$ points. By the above reasoning $2n^2$ lie at O, $n(2n+1)$ at ω, and $n(2n+1)$ at ω'. The $2n(n-1)$ remaining intersections are the ends of the chords of the $2n$-ic through O with length $2k$.

† But they may each be distinct algebraic curves; for instance, in the case of a circle.

By § 4 for the pedal with respect to O

$$n' = 2m, \quad m' = n + 2m, \quad \delta' = \{\tfrac{3}{2}m(m-1)\} + \{\tau\},$$
$$\kappa' = \{0\} + \{\iota\},$$

O, ω, ω' being m-ple points of this pedal.

Here in the expressions for δ' and κ' the term in the first brackets $\{\ \}$ refers to the multiple points at O, ω, ω' and the term in the second brackets $\{\}$ refers to multiple points elsewhere; and so in what follows.

For the conchoid of the first pedal we have by § 9

$$n' = 4m, \quad m' = 2n + 4m,$$
$$\delta' = \{m(6m-5)\} + \{m(m-1) + 2\tau\}, \quad \kappa' = \{2m\} + \{2\iota\}.$$

For this conchoid O, ω, ω' are $2m$-ple points, the tangents at ω, ω' coinciding in pairs.

For the inverse of this conchoid we have by § 3

$$n' = 2m, \quad m' = 2n + 2m, \quad \delta' = \{0\} + \{m(m-1) - 2\tau\},$$
$$\kappa' = \{0\} + \{2\iota\}.$$

For this inverse $O\omega$ and $O\omega'$ are m-ple tangents.

The polar reciprocal with respect to O of the inverse is the first negative pedal of the conchoid, i. e. the curve parallel to the given curve.

For the parallel curve we have finally

$$n' = 2n + 2m, \quad m' = 2m, \quad \tau' = m(m-1) - 2\tau, \quad \iota' = 2\iota,$$

ω and ω' being m-ple points of the parallel curve.

That the class of the parallel curve is twice that of the original curve is also evident from the fact that to each tangent to the given curve in a given direction correspond two parallel tangents of the parallel curve at a distance k from it. Similarly, to each inflexion of the given curve correspond two inflexions on the parallel curve.

If $\lambda x + \mu y + 1 = 0$ is a tangent to a curve with tangential equation $f(\lambda, \mu) = 0$, then

$$\lambda x + \mu y + 1 + k(\lambda^2 + \mu^2)^{\frac{1}{2}} = 0$$

is a tangent to the parallel curve, whose tangential equation is therefore

$$f\left(\frac{\lambda}{1 + k(\lambda^2 + \mu^2)^{\frac{1}{2}}}, \frac{\mu}{1 + k(\lambda^2 + \mu^2)^{\frac{1}{2}}}\right) = 0.$$

This equation is rationalized by a process similar to that used to obtain the polar equation of a conchoid in § 9. We leave to the reader the investigation of the properties of parallel curves by this means.

The parallel curve can readily be shown to be the envelope of a circle of radius k whose centre lies on the given curve. This gives in practice a convenient method of drawing curves parallel to a given curve.

Ex. 1. Show that the foci of a curve are singular foci of all parallel curves.

[Use the tangential equation.]

Ex. 2. Find the tangential equation of curves parallel to the ellipse

$$x^2/a^2+y^2/b^2 = 1.$$

Draw the parallel curves distinguishing the cases

$$k<b^2/a, \quad k=b^2/a, \quad b>k>b^2/a, \quad k=b, \quad a>k>b, \quad k=a,$$
$$a^2/b>k>a, \quad k=a^2/b, \quad k>a^2/b.$$

[$\{(a^2-k^2)\lambda^2+(b^2-k^2)\mu^2-1\}^2 = 4k^2(\lambda^2+\mu^2)$.]

Ex. 3. Find the tangential and parametral equations of curves parallel to the parabola $y^2=4ax$, and draw them.

[$(a\mu^2-\lambda)^2 = k^2\lambda^2(\lambda^2+\mu^2)$. Also, if $2\tan^{-1}t$ is the angle which a tangent to the parabola makes with the tangent at the vertex,

$$x=\frac{4at^2}{(1-t^2)^2}+\frac{k(1-t^2)}{1+t^2}, \qquad y=\frac{4at}{1-t^2}-\frac{2kt}{1+t^2}.$$

Changing the sign of k is equivalent to changing t into $-1/t$.

The curve is the reciprocal of the quartic of Ch. XVII, § 8 (IV). It has degree 6 and class 4. It has two linear branches osculating each other at $(\infty, 0)$, the conics of closest contact being $(y\pm k)^2=4ax$. It has also a node where $(k+2a)t^2=k-2a$, and six cusps where

$$2a(1+t^2)^3=k(1-t^2)^4.$$

It has no inflexion. It passes through the circular points, its singular focus being the focus of the parabola.

Distinguish the cases $k>2a$, $k=2a$, $k<2a$.

Discuss similarly $ay^2=x^3$ and $a^2y=x^3$.]

Ex. 4. The radii of a singly infinite family of circles are all increased by a constant. Show that their new envelope is parallel to their old one.

Ex. 5. Find the envelope of a family of circles whose centres lie on a given curve and which touch a given circle.

[Combine § 4, Ex. 18 and § 10, Ex. 4.]

Ex. 6. A rhombus *ABCD* with centre *O* moves in its own plane so that the lines through *O* parallel to the sides both touch a fixed curve. Show that *O* traces out an isoptic locus of the curve while *A*, *B*, *C*, *D* trace out the isoptic locus of a parallel curve; and that the normals at *O*, *A*, *B*, *C*, *D* to the two isoptic loci are concurrent.

§ 11. Other derived Curves.

Many other cases of derived curves might be given. But those already discussed must suffice. Further illustrations of the principle at the end of § 1 will be found in the following examples. The attention of the reader is especially called to Ex. 1 to 4, which will be found useful later on. The Plücker's numbers of the given curve are denoted in the examples by n, m, δ, κ, τ, ι, and those of the derived curve under consideration by n', m', δ', κ', τ', ι'.

Ex. 1. Find the Plucker's numbers of the locus of the centre of a circle touching a given curve and orthogonal to a given circle with centre O.

[We consider the case in which the given curve is of even degree n, having a $\frac{1}{2}n$-ple point at ω and ω'. The reader may consider other cases.*

To each of the n points in which a given line through O meets the given curve corresponds a tangent to the locus perpendicular to the line. Hence $m' = n$.

To each tangent from O to the given curve corresponds an intersection of the locus with $\omega\omega'$. Hence $n' = m$.

To each cusp of the given curve corresponds an inflexion of the locus. Hence $\iota' = \kappa$.

We see that the locus has the same Plucker's numbers as the reciprocal of the given curve.]

Ex. 2. How many circles can be drawn orthogonal to a given circle and bitangent to or osculating a given curve?

[Such a circle is given by each node or cusp of the locus of Ex. 1.]

Ex. 3. Find the Plücker's numbers of the envelope of a circle with its centre on a given curve s and cutting orthogonally a given circle j whose centre is O.

$$[n' = 2m, \qquad m' = 2(n+m), \qquad \delta' = m(m-1)+2\tau, \qquad \kappa' = 2\iota,$$
$$\tau' = n^2+4mn+2m^2-10m+2\delta, \qquad \iota' = 2(3m+\kappa), \qquad D' = (m-1)+2D.$$

The envelope is the locus of the points Q_1, Q_2 on the perpendicular OY from O to the tangent at any point P of s, which are equidistant from Y and inverse for j. The envelope is self-inverse with respect to j, and has an m-ple point at ω and ω'. The tangents from O to the envelope are n bitangents given by taking P at infinity and the $2m$ perpendiculars from O to the $2m$ common tangents of j and s. Each bitangent of s gives two nodes of the envelope and each inflexion gives two cusps. The foci of s are the singular foci of the envelope, and the $2n$ intersections of j and s are ordinary foci of the envelope.

The circle with centre Y and radius equal to the tangents YL and YL' from Y to j passes through Q_1 and Q_2, while PQ_1 and PQ_2 are the normals at Q_1 and Q_2 to the envelope. If OR is perpendicular to LL', P and R are corresponding points on s and its polar reciprocal with respect to j. These facts enable us to construct any number of points on the envelope, when either s or its polar reciprocal with respect to j are drawn.

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* If the curve has a k-ple point at each of ω and ω', $n' = m+2n-4k$, $m' = 2n-2k$, $\iota' = \kappa$.

If j is $x^2+y^2 = k$ and s has the tangential equation

$$0 = a+f_1(\lambda, \mu)+f_2(\lambda, \mu)+f_3(\lambda, \mu)+ \ldots,$$

where $f_r(\lambda, \mu)$ is homogeneous of degree r in λ and μ, the equation of the envelope is

$$0 = a-2f_1(x, y)/(x^2+y^2+k)+4f_2(x, y)/(x^2+y^2+k)^2$$
$$-8f_3(x, y)/(x^2+y^2+k)^3+ \ldots.]$$

Ex. 4. What modification must be made in the results of Ex. 3 if the curve s (i) touches $\omega\omega'$ at H; (ii) goes through ω and ω'; (iii) touches the tangents from a focus S at two points on j?

[(i) The envelope cuts OH orthogonally at O and has an asymptote perpendicular to OH twice as far from O as the corresponding asymptote of the pedal of s with respect to O. See § 4, Ex. 11.

(ii) The tangents at ω and ω' to s are cuspidal tangents of the envelope.

(iii) $S\omega$ and $S\omega'$ are inflexional tangents of the envelope.]

Ex. 5. Find the locus of the middle point of a chord of a given curve drawn in a fixed direction.

[$n' = \frac{1}{2}n(n-1)$, $m' = (n-2)m$, $\kappa' = (n-2)\kappa$. Consider the infinite points on the locus and the tangents in the fixed direction.]

Ex. 6. How many lines can be drawn in a given direction on which a given n-ic intercepts $\frac{1}{2}n(n-1)$ segments two of which have the same middle point?

[In Ex. 5 $\delta' = \frac{1}{8}n(n-1)(n-2)(n-3)+(n-2)\delta$. Of these nodes of the locus $(n-2)\delta$ are given by the nodes of the given n-ic, and the others by the segments in question.]

Ex. 7. Find the Plücker's numbers of the locus of the middle point of the segment intercepted by two fixed lines on any tangent to a given curve.

[$n' = 2m$, $m' = 2m+n$, $\kappa' = \iota$.]

Ex. 8. Find the degree of the locus of the end of the polar subnormal of a given curve.

[$2n+m$. Consider the intersections of the locus with the line at infinity.]

Ex. 9. Find the degree of the *radial* of a given curve; i.e. the locus of the end of the line through a fixed point O parallel and equal to the radius of curvature at each point of the curve.

[$3m+\kappa$. The cusps and points of contact of tangents from ω and ω' give branches of the radial through O. Any line through O meets the radial in m more points. Prove also $m' = 7m-3n+3\kappa$, $\kappa' = n$, $D' = D$.]

Ex. 10. Find the degree of the locus of the centre of a circle touching a given curve and a given straight line.

[$2(m+n)$. The locus meets the given line at its intersections with the tangents to the curve from ω and ω', and has a node at each intersection of the given line with the curve.]

Ex. 11. The tangent at P to a curve meets it again in Q, R, ... Find the degree of the locus of X if (i) X divides PQ in a given ratio, (ii) X divides QR in a given ratio.

[It is easy to find the intersections of the loci with the line at infinity. Their degrees are

$$n(m+2n-6) \quad \text{and} \quad 2(m-2)n(n-3).]$$

Ex. 12. Find the degree of the locus of X, if the tangent at P is replaced by the normal at P in Ex. 11.

Ex. 13. Find the degree of the locus of the intersection of two equal tangents to a curve.

[The circular points are $\frac{1}{2}m(m-1)$-ple points of the locus, and for the other points at infinity see § 2, Ex. 6.

Discuss the locus of a point from which equal tangents can be drawn to two different curves.]

Ex. 14. If P is any variable point on the first polar curve of a fixed point O with respect to a given curve, the locus of intersection of the polar line of P and the polar of P with respect to a fixed conic is a curve of degree $m+\delta+\kappa$.

[The polar line of P passes through O which is a $(n-1)$-ple point of the locus. Find where the polar line meets the locus again.]

CHAPTER XII

INTERSECTIONS OF CURVES

§ 1. Conditions determining a Curve.

THE equation $f(x, y, z) = 0$ of a curve of degree n contains one term of zero degree in x and y, two terms of the first degree, ..., $n+1$ terms of the n-th degree. Hence $f(x, y, z)$ contains $\frac{1}{2}(n+1)(n+2)$ coefficients.

Suppose we are given r points

$$(x_1, y_1, z_1), \quad (x_2, y_2, z_2), \quad \ldots, \quad (x_r, y_r, z_r).$$

The curve will pass through them if

$$f(x_1, y_1, z_1) = 0, \ f(x_2, y_2, z_2) = 0, \ \ldots, \ f(x_r, y_r, z_r) = 0. \quad \text{(i)}$$

These are linear equations in the $\frac{1}{2}(n+1)(n+2)$ coefficients of $f(x, y, z)$. They will determine the

$$\tfrac{1}{2}(n+1)(n+2) - 1 = \tfrac{1}{2}n(n+3)$$

independent ratios of the coefficients, if

$$r = \tfrac{1}{2}n(n+3).$$

Hence in general one and only one curve of degree n passes through $\frac{1}{2}n(n+3)$ given points (see Ch. II, § 6).

It may happen, however, that the equations (i) are not independent. In this case an infinite number of n-ics will pass through the $\frac{1}{2}n(n+3)$ given points.

Let us consider two special cases.

First take $n = 1$. In this case $\frac{1}{2}n(n+3) = 2$. Equations (i) have one and only one solution; for one and only one straight line passes through two given points.

Next take $n = 2$. Here $\frac{1}{2}n(n+3) = 5$.

If no four of the five assigned points are collinear, one and only one conic goes through them.

If three of the five points are collinear, the conic through the points is degenerate, being the line through these three points and the line joining the other two points.

If four of the given points are collinear, there are a singly infinite number of conics through the five points, namely the line through the four points and any line through the fifth point.

Lastly, if all five given points are collinear, there are a doubly infinite number of conics through the five points, namely the line through the points and any other line whatever.

If $r<\frac{1}{2}n(n+3)$, equations (i) express the ratios of the coefficients of $f(x, y, z)$ linearly in terms of $\frac{1}{2}n(n+3)-r$ arbitrary independent quantities; so that an $\{\frac{1}{2}n(n+3)-r\}$-ply infinite number of n-ics pass through the given r points in general. If only s of equations (i) are independent, the ratios of the coefficients can be expressed linearly in terms of $\frac{1}{2}n(n+3)-s$ quantities. The equation of the n-ic is then of the form

$$k_1w_1+k_2w_2+\ldots+k_sw_s=0;$$

where $k_1, k_2, \ldots, k_s$ are arbitrary constants, and

$$w_1=0,\ w_2=0,\ \ldots,\ w_s=0$$

are s fixed n-ics through the given points.

Ex. If p and q are positive integers such that a p-ic and a q-ic can be described through any pq arbitrary points, we must have $p=q=1$, $p=1$ and $q=2$, $p=2$ and $q=1$, or $p=q=2$.

[These are the only values of p and q such that

$$\tfrac{1}{2}p(p+3)\geqslant pq \quad\text{and}\quad \tfrac{1}{2}q(q+3)\geqslant pq.]$$

§ 2. Cubics through Eight Points.

We now take the case $n=3$. Here $\frac{1}{2}n(n+3)=9$.

Suppose eight points on the cubic are given. The coefficients of its equation can in general be expressed linearly in terms of a single quantity. Hence the equation is of the form $u+kv=0$, where $u=0$ and $v=0$ are fixed cubics, and k is a parameter not involving x, y, z. Therefore the cubic passes through the nine fixed intersections of $u=0$ and $v=0$; eight of them being the eight given points. Hence:

All cubics through eight fixed points pass through a ninth fixed point.

In general one and only one cubic passes through nine given points. But if the nine points are the intersections of two cubics, an infinite number of cubics pass through the nine given points.*

§ 3.

It is necessary now to consider whether there is any exception to the statement that all cubics through eight points pass through a ninth. The argument which established this result will not be valid if the eight points are such that all cubics through any seven of them necessarily pass through the eighth; which implies that the equations (i) of § 1 ($n = 3, r = 8$) are not independent. The equation of the cubics will involve not one, but two or more parameters in this case; and the preceding proof breaks down in consequence.

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* In this case any two of the nine points are said to be conjugate to one another with respect to the other seven.

Let the eight points then be A, B, C, D, E, F, G, H, which are such that any cubic through seven of them passes through the eighth.

First suppose no three of the eight points collinear. The cubic consisting of the conic $ABCDE$ and the line GH passes through seven of the points and therefore through the eighth point F. But F does not lie on GH. Hence the conic $ABCDE$ goes through F, and similarly through G and H. The eight given points lie on a conic, and any cubic through them consists of this conic and some straight line; for a non-degenerate cubic cannot meet a conic in more than six points.

The case in which three or more of the given points are collinear is dealt with in a similar manner. It will be found that the eight points lie on one or other of two straight lines, one (or both) of which is part of every cubic through the eight points.

Hence all cubics through eight given points always pass through a ninth fixed point, and may under special circumstances pass through an infinite number of other fixed points.

§ 4.

Let us now consider some applications of the theorem of § 2.

If two straight lines meet a cubic in A_1, A_2, A_3 *and* B_1, B_2, B_3, *while the lines* A_1B_1, A_2B_2, A_3B_3 *meet the cubic again in* C_1, C_2, C_3, *then* C_1, C_2, C_3 *are collinear.*

For the three cubics

(1) the given cubic,

(2) the line-trio $A_1A_2A_3$, $B_1B_2B_3$, C_1C_2,

(3) the line-trio $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$,

pass through the eight points $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$. They therefore pass through a ninth point. But cubics (1) and (3) pass through C_3. Hence C_3 lies on cubic (2). But C_3 cannot lie on the lines $A_1A_2A_3$ or $B_1B_2B_3$, since neither of them meets the cubic in four points. Therefore C_3 lies on C_1C_2 (Fig. 1).

Taking the lines $A_1A_2A_3$, $B_1B_2B_3$ very close to one another, and defining the *tangential* of a point A on a cubic as the point in which the tangent at A meets the curve again, we have:

The tangentials of three collinear points of a cubic are collinear.

Suppose now A_1 and A_2 are inflexions of the cubic, while $A_1A_2A_3$ and $B_1B_2B_3$ are close to one another. Then C_1C_2 is close to them, and we have in the limit:

*The line joining two inflexions of a cubic passes through another inflexion.**

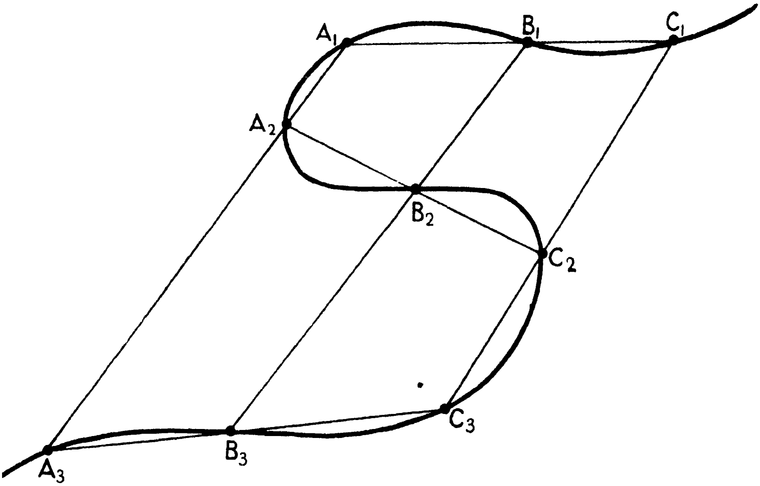

Fig. 1.

Ex. 1. The finite intersections of a cubic with its asymptotes are collinear.

[They are the tangentials of its intersections with the line at infinity. See Ch. II, § 5, Ex. 3.]

Ex. 2. The points of contact of the tangents from an inflexion of a cubic are collinear.

[This is a particular case of the collinearity of tangentials of collinear points. The points of contact lie, of course, on the harmonic polar of the inflexion.]

Ex. 3. If A, B, C, D, E, F are points on a conic, while AB and DE meet at L, BC and EF at M, CD and FA at N, then L, M, N are collinear.

[Consider (1) conic and line LM, (2) line-trio AB, CD, EF, (3) line-trio BC, DE, FA. This is Pascal's well-known theorem.]

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* See also § 5, Ex. 9; Ch. II, § 3, Ex. 7; Ch. VII, § 4.

Ex. 4. If six of the intersections of two cubics lie on a conic, the other three are collinear.

[Consider the two given cubics and the cubic consisting of the conic and the line joining two of the remaining intersections. The theorems in § 4 and many of these examples are particular cases of this result.]

Ex. 5. If two cubics have the same asymptotes, their finite intersections are collinear.

[The conic of Ex. 4 is the line at infinity twice over. Ex. 1 is a particular case.]

Ex. 6. A conic meets a cubic at A, B, C, D, E, F. Show that AB, CD, EF meet the cubic again at three collinear points L, M, N.

[The second cubic of Ex. 4 is the line-trio ABL, CDM, EFN.]

Ex. 7. A conic touches a cubic at P, Q, R. Show that the tangentials of P, Q, R are collinear.

Show that there are three families of conics, each doubly infinite in number, having triple contact with a given non-singular cubic.

Discuss the case of a unicursal cubic.

[See Ex. 6. P and Q may be chosen arbitrarily. When they are chosen, the tangentials of P, Q, R are known. Then there are three possible positions for R, excluding the case in which P, Q, R are collinear.]

Ex. 8. A conic osculates a cubic at P and Q. Show that PQ passes through an inflexion.

[See Ex. 6.]

Ex. 9. The sextactic points of a cubic (points at which a conic has six-point contact) are the points of contact of the tangents from the inflexions. Discuss the number of such sextactic points in the cases of a non-singular, nodal, or cuspidal cubic.

[A particular case of Ex. 8. The number of points is 27, 3, 0.]

Ex. 10. A circle meets a circular cubic in the finite points A, B, C, D. If AB and CD meet the cubic again in L and M, LM is parallel to the real asymptote of the cubic.

[In Ex. 6 take E and F at the circular points.]

Ex. 11. A conic has four-point contact with a cubic at A and meets the curve again at E and F. If EF meets the cubic again in N, N is the tangential of the tangential of A.

[Take A, B, C, D consecutive in Ex. 6.]

Ex. 12. Enunciate the theorem obtained by making A and B, C and D respectively consecutive in Ex. 6.

Ex. 13. The tangents from a point P of a cubic are PA, PB, PC, PD. Show that AB and CD meet at Q on the curve.

[Consider (1) the given cubic, (2) polar conic of P and line AB, (3) line-trio PA, PB, CD.]

Ex. 14. Show that the tangents at P and Q in Ex. 13 meet on the curve.

[The tangentials of A, Q, B are collinear.]

Ex. 15. Two cubics go through the vertices A, B, C, D of a quadrangle and through the diagonal points E, F, G. Show that, if they touch at A, they osculate there; while, if they touch at E, they meet again on FG.

[Consider (i) two cubics and line-trio ABE, ACG, ADF; (ii) two cubics and line-trio EBA, ECD, FG.]

Ex. 16. Chords PQ, RS of a cubic meet at T on the curve. Show that a quadrilateral can be inscribed in the cubic with the sides AB, BC, CD, DA going through P, R, Q, S and any given point A of the curve as vertex.

[Consider (1) given cubic, (2) line-trio SRT, APB, CQD, (3) line-trio PQT, BRC, DSA.]

Ex. 17. A conic passes through four fixed points A, B, C, D of a cubic. Show that the line joining the other two intersections of the conic and cubic meets the cubic again at a fixed point.

[Let conics S, S' through A, B, C, D meet the cubic again in E, F and E', F'. Consider the cubics (1) given cubic, (2) conic $ABCDEF$ and line $E'F'$, (3) conic $ABCDE'F'$ and line EF.]

Ex. 18. Through four given points of a cubic four conics can be drawn touching the cubic at some other point. The tangents at the four points of contact all meet on the cubic.

[See Ex. 17.]

Ex. 19. The circle of curvature at A to a circular cubic passes through B, and the circle of curvature at B passes through A. Show that the tangents at A and B meet on the cubic.

[Take C, D in Ex. 17 at the circular points.]

Ex. 20. Two points P, P' on a cubic are joined to points A, B on the curve; and PA, PB, $P'A$, $P'B$ meet the curve again at Q, R, Q', R'. Prove that $Q'R$ and QR' meet on the cubic.

[See Ex. 17.]

Ex. 21. A variable point P on a cubic is joined to two fixed points of the curve, and the joining lines meet the curve again in Q and R. Prove that QR is divided harmonically by the cubic and its point of contact with its envelope.

[Make P' consecutive to P in Ex. 20.]

Ex. 22. Any cubic is the locus of the intersections of a pencil of conics with a homographic pencil of lines whose vertex is any given point O of the cubic.

[Let ABC be any three fixed points of the cubic. If any line through O meets the 3-ic in O, P, Q, the conic $ABCPQ$ meets the 3-ic again in a fixed point D (Ex. 17) and the line OPQ and the conic $ABCDPQ$ have a one-to-one correspondence. The point O is called *the point opposite to* A, B, C, D with respect to the cubic.]

Ex. 23. Prove the accuracy of the following ruler construction for the point opposite to A, B, C, D with respect to the cubic through $A, B, C, D, E, F, G, H, I$. 'Let the conics through A, B, C, D be put into homographic correspondence with the lines through H so that the lines HE, HF, HG correspond respectively to the conics $ABCDE$. $ABCDF$, $ABCDG$. Let the line through H corresponding in the homography to the conic $ABCDI$ meet in I' the conic Σ through E, F, G, H touching at H the line corresponding to the conic $ABCDH$. Then HI' meets Σ again at O.'

Ex. 24. Show that if O and Ω are the points opposite to A, B, C, D and to E, F, G, H, the conics $ABCD\Omega$ and $EFGHO$ meet $O\Omega$ at the ninth common point of all cubics through A, B, C, D, E, F, G, H.

Deduce a ruler construction for the ninth intersection of all cubics through eight given points.

Ex. 25. Given nine points, construct by ruler the intersection of the cubic through them with the line through two of the points or the conic through five.

§ 5. Intersections of Two n-ics.

The theorem that all cubics through eight given points pass through a ninth fixed point can be extended to curves of higher degree in the form:

In general all curves of the n-th degree through $\frac{1}{2}n(n+3)-1$ fixed points pass through $\frac{1}{2}(n-1)(n-2)$ other fixed points.

For in general by § 1 any such curve has an equation of the form $u+kv=0$, where k is a parameter; and the curve passes through all the n^2 intersections of $u=0$ and $v=0$. Noting that

$$\tfrac{1}{2}n(n+3)-1+\tfrac{1}{2}(n-1)(n-2)=n^2,$$

we have the required result.

The theorem is only true 'in general' if $n>3$; i.e. only if the given points are such that not all n-ics through $\frac{1}{2}n(n+3)-2$ of them pass through the remaining point. We shall assume that the theorems of §§ 5, 6, 7 hold good in the examples given in these sections; but it must be admitted that this assumption is hardly rigorous without further investigation.

Ex. 1. A conic meets a quartic at A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, and the lines A_1A_2, B_1B_2, C_1C_2, D_1D_2 meet the quartic again at A_3 and A_4, B_3 and B_4, C_3 and C_4, D_3 and D_4. Show that A_3, A_4, B_3, B_4, C_3, C_4, D_3, D_4 lie on a conic.

[(1) The given quartic, (2) the lines A_1A_2, B_1B_2, C_1C_2, D_1D_2. (3) the given conic and the conic $A_3B_3C_3D_3A_4$, are three quartics through the same thirteen points. Therefore they have three more points in common. Ex. 2, 3, 4 are special cases of this result.]

Ex. 2. The tangents at four collinear points of a quartic meet the curve again in eight points on a conic.

Ex. 3. Deduce Ch. II, § 3, Ex. 8, 9, 10 from Ex. 1.

Ex. 4. Four normals are drawn to a conic from a point, and from the centres of curvature at the feet of the normals two more normals are drawn. Show that these eight normals touch a conic.

[Reciprocate and use Ex. 2.]

Ex. 5. A straight line meets a quartic at A, B, C, D. Show that the twelve points in which a conic through A, B and a conic through C, D meet the quartic again lie on a cubic.

[Consider the quartics (1) given quartic, (2) pair of conics, (3) given line and cubic through nine of the twelve points.]

Ex. 6. If a conic passes through the points of contact of a bitangent of a quartic, a cubic will touch the quartic at its six other intersections with the quartic.

[Limiting case of Ex. 5.]

Ex. 7 Through each of n collinear points of an n-ic a straight line is drawn. Show that they meet the n-ic again in $n(n-1)$ points on an $(n-1)$-ic.

[Consider the n-ics (1) given n-ic, (2) n lines, (3) line of collinear points and $(n-1)$-ic through $\frac{1}{2}(n-1)(n+2)$ of the $n(n-1)$ points.]

Ex. 8. A conic meets an n-ic in $2n$ points. These are divided into pairs. Show that the n lines joining each pair meet the n-ic again in $n(n-2)$ points on an $(n-2)$-ic.

[Consider the n-ics (1) given n-ic, (2) n lines, (3) conic and $(n-2)$-ic. This includes Ex. 1 as a special case. Generalize Ex. 2.]

Ex. 9. A line through $n-1$ inflexions of an n-ic passes through an n-th inflexion.

[Take the conic of Ex. 8 as the line twice over and the n lines as the tangents at its intersections with the n-ic. The result also follows from Ch. I, § 6, Ex 9. The case $n = 3$ is well known.]

Ex. 10. All n-circular $2n$-ics through n^2+2n-1 finite points pass through $(n-1)^2$ other fixed finite points

Ex. 11. Show that the theorem of § 5 is not *always* true when $n > 3$.

[If two quartettes of lines meet in sixteen points of which three lie on another line, the theorem is readily seen to be not true for 4-ics through the other thirteen points. More generally, it may be shown to be untrue for the thirteen other intersections of any two 4-ics through three collinear points.]

Ex. 12. The feet of the normals to a curve $f(x, y) = 0$ of degree n and class m from a point $O(\xi, \eta)$ lie on an n-ic through O and the poles of the line at infinity, the directions of whose asymptotes are the axial directions of $f=0$. The number of such normals is $m+n$.

[On $(x-\xi)\dfrac{\partial f}{\partial y} = (y-\eta)\dfrac{\partial f}{\partial x}$, which meets $f=0$ twice at each node and thrice at each cusp. See Ch. VII, § 2, Ex. 13]

Ex. 13. Any n-ic through the feet of $\frac{1}{2}n(n+3)-1$ normals from O to a given n-ic passes through the foot of every normal from O $(n > 2)$.

§ 6.

In general, if np of the n^2 intersections of two n-ics lie on a p-ic, the remaining $n(n-p)$ lie on an $(n-p)$-ic.

For the p-ic and an $(n-p)$-ic through $\frac{1}{2}(n-p)(n-p+3)$ of the remaining points form an n-ic (degenerate) through

$$np+\tfrac{1}{2}(n-p)(n-p+3) = \tfrac{1}{2}n(n+3)-1+\tfrac{1}{2}(p-1)(p-2)$$

of the n^2 intersections of the two given n-ics. Hence by § 5 all the remaining points lie on this degenerate n-ic. But none of them can lie on the p-ic, for the p-ic cannot meet the n-ic in more than np points unless the n-ic degenerates, which we suppose not to be the case. Hence they must all lie on the $(n-p)$-ic.

Ex. 1. A polygon of $2n$ sides is inscribed in a conic. The intersections of the sides, other than the vertices of the polygon, lie on an $(n-2)$-ic.

[Consider the two n-ics formed by taking every alternate side of the polygon: $2n$ of their intersections lie on a conic.
Note the cases $n = 3$ or 4.]

Ex. 2. Any quartic through the intersections of a cubic and quartic meets the quartic again in four collinear points.

Ex. 3. Any quartic through the intersections of a conic and quartic meets the quartic again in eight points on a conic.

Ex. 4. A conic meets a quartic in P, Q, R, S, P', Q', R', S'. Any conic through P, Q, R, S and any conic through P', Q', R', S' meet the quartic again in eight points on a conic.

[This and the following examples are particular cases of Ex. 3. Obtain other theorems as special cases]

Ex. 5. Two conics are drawn through the points of contact of two bitangents of a quartic. Show that their eight other intersections with the quartic lie on a conic.

[See also Ch. XIX, § 2, Ex. 2.]

Ex. 6. A conic through the points of contact of two bitangents of a quartic meets the curve again in A, B, C, D. Show that a conic can be drawn touching the quartic at A, B, C, D.

Ex. 7. If a conic meets a quartic in eight points and a conic touches the quartic at four of the points, a conic touches the quartic at the other four.

§ 7. Intersections of any Two Curves.

An extension of the theorem of § 5 is the following:

In general any r-ic through all but

$$\tfrac{1}{2}(n+N-r-1)(n+N-r-2)$$

of the nN intersections of an n-ic and N-ic will pass through the remaining intersections, provided $r \geqslant n$, $r \geqslant N$, $r \leqslant n+N-3$.*

If $n = N = r$, we have the theorem of § 5.

Take $\tfrac{1}{2}(r-n)(r-n+3)$ arbitrary points on the N-ic and $\tfrac{1}{2}(r-N)(r-N+3)$ arbitrary points on the n-ic.

Take also

$$nN-\tfrac{1}{2}(n+N-r-1)(n+N-r-2)$$

of the intersections of n-ic and N-ic.

---

* For a discussion of limitations to which this result is subjected, see Bacharach, *Math. Annalen* xxvi (1886), pp. 275–299.

The total number of points taken will be found to be

$$\tfrac{1}{2}r(r+3)-1.$$

The n-ic together with the $(r-n)$-ic through the

$$\tfrac{1}{2}(r-n)(r-n+3)$$

points form a degenerate r-ic through the $\frac{1}{2}r(r+3)-1$ points; and so for the N-ic together with the $(r-N)$-ic through the $\frac{1}{2}(r-N)(r-N+3)$ points. Therefore every r-ic through the $\frac{1}{2}r(r+3)-1$ points passes through every intersection of this pair of degenerate r-ics; which proves the result.

Ex. 1. If a quartic meets a cubic in $A, B, C, D, E, F, A', B', C', D', E', F'$, the lines $AA', BB', CC', DD', EE', FF'$ meet the cubic again in six points on a conic.

[n-ic is given cubic, N-ic is the quartic and the conic through five of the points, r-ic is the six lines.]

Ex. 2. The tangents at the six intersections of a conic and cubic meet the cubic again in six points on a conic.

[Take the quartic in Ex. 1 as a pair of adjacent conics.]

Ex. 3. The tangents from a point O to a cubic meet the curve again in six points on a conic.

[Take the conic of Ex. 2 as the polar conic of O. See Ch. I, § 9, Ex. 6.]

Ex. 4. A quartic through the intersections of two cubics meets either cubic again in three collinear points.

[n-ic is cubic, N-ic is quartic, r-ic is other cubic and the line through two of the three points.]

§ 8. Theory of Residuals.

Suppose that C_r denotes a homogeneous expression of degree r in x, y, z. The curve $C_r=0$ is a curve of degree r, which we shall call 'the curve r', when there can be no confusion with the number r.

Suppose $C_l, C_\lambda, C_m, C_\mu, C_n, C_\nu$ to be such that

$$C_l C_\lambda + C_m C_\mu + C_n C_\nu \equiv 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

where $l+\lambda = m+\mu = n+\nu$.

Suppose that the curves l, m, n all pass through points which we call collectively 'the points O'. Let the complete intersection of the curves m and n be the points O together with other points which we call collectively 'the points L'. Then (i) shows that the curve λ passes through L.

Suppose that

the curves l and m meet in the points O and N,
the curves l and n meet in the points O and M,
the curves λ and m meet in the points L and M',
the curves λ and n meet in the points L and N'.

By (i) the curves μ and ν meet in points each of which lies on one of the curves l or λ. Suppose they meet in points L' on l and O' on λ. Now by (i) the curves l and m meet in points each of which lies on one of the curves n or ν. But the only points common to the curves l, m, n are O; so that the curve ν passes through the points N.

Proceeding thus, we obtain relations between the curves and their intersections which will be clear from Fig. 2. In this diagram is shown a cube with unit edge whose faces

$$x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$$

represent the curves l, λ, m, μ, n, ν respectively.

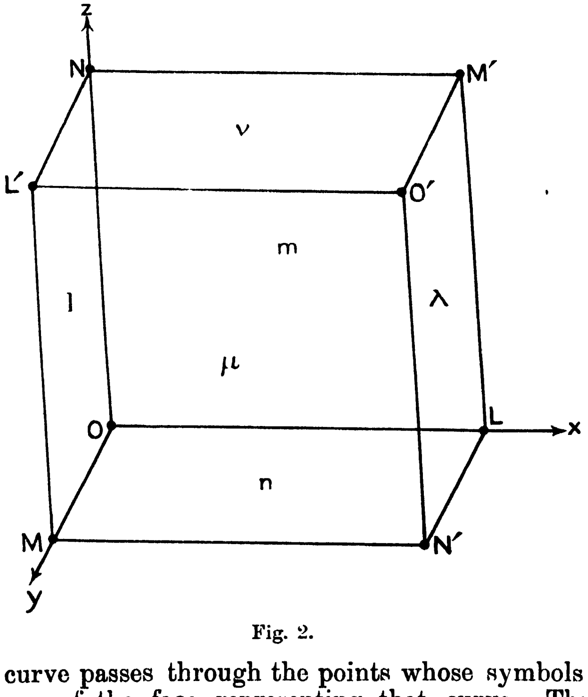

Fig. 2.

Each curve passes through the points whose symbols lie at the corners of the face representing that curve. Thus the curve l passes through the points $OL'MN$, the curve λ passes through the points $O'LM'N'$, and so on. Moreover, the complete intersection of any two curves represented by faces of the cube (not opposite) is the points whose symbols lie at the ends of the edge common to those faces. Thus OL is the complete intersection of the curves m and n, MN' of μ and n, $O'L'$ of μ and ν, &c.

Ex. 1. The n-ics through any given point O and the intersections of three given n-ics taken in pairs form a pencil.

[Let $f=0$, $\phi=0$, $\psi=0$ be the given n-ics; f_0, ϕ_0, ψ_0 what f, ϕ, ψ become when we substitute in them the coordinates of O. Then use

$$f_0(\phi_0\psi-\psi_0\phi)+\phi_0(\psi_0 f-f_0\psi)+\psi_0(f_0\phi-\phi_0 f)\equiv 0.]$$

Ex. 2. Through the nN intersections of an n-ic and an N-ic three n-ics are drawn. Through the remaining intersections of each pair of n-ics is drawn an $(n-N)$-ic. Show that these three $(n-N)$-ics form a pencil.

[If $C_N=0$ and $C_n=0$ are the N-ic and n-ic, the three n-ics are

$$C_n+C_N\mathrm{A}=0, \quad C_n+C_N\mathrm{B}=0, \quad C_n+C_N\Gamma=0,$$

and the $(n-N)$-ics are $\mathrm{B}=\Gamma$, $\Gamma=\mathrm{A}$, $\mathrm{A}=\mathrm{B}$.]

§ 9.

Let us now consider $C_n=0$ as a given n-ic with any points O taken on it. We assume that, subject to certain limitations as to the positions of the points O,* &c., it is always possible to find polynomials C_μ, C_ν such that

$$C_l C_\lambda + C_m C_\mu + C_n C_\nu \equiv 0;$$

where C_l, C_m, C_λ are given polynomials such that $l+\lambda$ is not less than m and n, while the curves $C_l=0$, $C_m=0$ meet at each of the points O, but at no other point of the given n-ic, and $C_\lambda=0$ passes through the intersections of $C_n=0$ and $C_m=0$ other than O.

For the proof of this important theorem see § 10.

We assume its truth and deduce some consequences.

As in § 8 we may consider O and M as the complete intersection of $C_l=0$ and $C_n=0$, and O and L as the complete intersection of $C_m=0$ and $C_n=0$. Suppose any curve $C_\lambda=0$ through L meets $C_n=0$ in the points L and N'. Then the points M and N' are the complete intersection of $C_n=0$ with some curve, namely $C_\mu=0$.

The points O and M are said to be *residual* groups of points for the curve $C_n=0$, meaning that, taken together, they are the complete intersection of $C_n=0$ with some curve ($C_l=0$). This will be expressed by the notation

$$O+M\equiv 0.\dagger$$

Similarly O and L are residual, i. e.

$$O+L\equiv 0,$$

since O and L form the complete intersection of $C_n=0$ with $C_m=0$.

---

* For instance, care is necessary if some of the points O are multiple points of the n-ic.

† And in general, if groups of points P, Q, R, ... form taken all together the complete intersection of C_n with a given curve, we write $P+Q+R+\ldots\equiv 0$.

The points L and M are said to be *coresidual* groups of points for the curve C_n, meaning that they are both residual to the same points O. We denote this by the notation

$$L \equiv M.$$

The result above obtained may then be worded:

If L and M are coresidual groups of points for an n-ic, any points residual to L are residual to M.

In fact, we proved that, since L and M were both residual to O, any points N' residual to L were residual to M. In other words, if OL, OM, LN' are all complete intersections of the n-ic with some other curve, so are MN'.

Symbolically, we deduce from

$$O+L \equiv 0,\ O+M \equiv 0,\ L+N' \equiv 0, \text{ that } M+N' \equiv 0.$$

Remembering the notation $L \equiv M$, we see that symbolical relations such as $O+L \equiv 0$, &c, can be added or subtracted just as if they were ordinary identities. It is only necessary to consider such a relation as $L-M \equiv 0$ to be equivalent to $L \equiv M$, and $2P+Q \equiv 0$ to mean $P+P+Q \equiv 0$,* &c.

Such a treatment of the symbolism is a great economy of labour. It enables us to some extent to replace geometrical reasoning by elementary algebraical work; though, of course, at some stage or other the algebraic result must be interpreted geometrically, if it is to be of any value.

We must be careful to confine our algebraic processes to addition and subtraction. Multiplication by an integer will be lawful, for that is only equivalent to repeated addition; but division is not allowed.

Ex. 1. Two single non-coincident points P, Q cannot be coresidual ($n > 2$).

[Any line through P other than PQ meets the n-ic in $n-1$ points which cannot form with Q the complete intersection of the n-ic with any curve.]

Ex. 2. Through points P on a given n-ic any curve is drawn meeting the n-ic again in points P_1, through P_1 is drawn any curve meeting the n-ic again in P_2, through P_2 is drawn any curve meeting the n-ic again in P_3, and so on. If P_{2r} consists of a single point, show that this point is the same whatever the curves used and whatever the value of r.

[$P+P_1 \equiv 0,\quad P_1+P_2 \equiv 0,\quad P_2+P_3 \equiv 0,\quad \dots,\quad P_{2r-1}+P_{2r} \equiv 0$ give $P \equiv P_{2r}$. Suppose other curves used; then we have $P \equiv P'_{2s}$ (say). Hence $P_{2r} \equiv P'_{2s}$. By Ex. 1 if P_{2r} and P'_{2s} are single points, they coincide.

If the points P are p in number, and the curves are of degrees

$n_1, n_2, \dots, n_{2r}$, we have $n(n_1-n_2+n_3-\dots-n_{2r})-p+1=0$.]

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* 'The points Q and the points P taken twice from the complete intersection of the n-ic with some curve.'

Ex. 3. Through four points of a given 3-ic draw a conic meeting the given 3-ic in two more points, through these points draw a 3-ic meeting the given 3-ic in seven more points, through them draw a 4-ic meeting the 3-ic in five more points, through them draw a conic meeting the 3-ic in one more point R. Show that any conic through the four points initially taken meets the 3-ic again in two points collinear with R.

[See Ex. 2. The reader may enunciate similar theorems.]

Ex. 4. Two pairs of points on an n-ic ($n > 3$) cannot be coresidual, if no three of the four points are collinear.

[As in Ex. 1.]

Ex. 5. Two trios of points on a quartic cannot be coresidual (unless they coincide).

[One trio is residual to the remaining five intersections of the 4-ic with any conic through the other trio. Therefore the trios coincide.]

Ex. 6. Through seven points P of a quartic a cubic is drawn meeting the quartic in five more points Q. Through P is drawn a quartic meeting the given quartic in nine more points R. Show that the conic through Q and the cubic through R meet in three points on the quartic.

[Use Ex. 5.]

Ex. 7. The tangents at n collinear points of an n-ic meet the curve again in $n(n-2)$ points on an $(n-2)$-ic.

[Denoting the n points by P and the $n(n-2)$ points by Q,

$$P \equiv 0 \quad \text{and} \quad 2P + Q \equiv 0.$$

Hence $Q \equiv 0$. For the case $n = 3$, see § 4. See also § 5, Ex. 8.]

Ex. 8. From any point O $n(n-1)$ tangents are drawn to a non-singular n-ic. Show that they meet the curve again in $n(n-1)(n-2)$ points lying on an $(n-1)(n-2)$-ic.

[Denoting the points of contact, which lie on the first polar of O, by P, and the $n(n-1)(n-2)$ points by Q, $P \equiv 0$ and $2P + Q \equiv 0$. Hence $Q \equiv 0$. For the case $n = 3$ see § 7, Ex. 3.]

Ex. 9. If six intersections of a 3-ic and 4-ic lie on a conic, so do the remaining six. The four other intersections of the 4-ic and the two conics are collinear.

[Denote the six intersections of the 4-ic and the first conic by P and the other two intersections by R. Let the line joining R meet the 4-ic again in two points S. Let the 3-ic and 4-ic meet in six other points Q. Then $P + Q \equiv 0$, $P + R \equiv 0$, $R + S \equiv 0$ give $Q + S \equiv 0$.]

Ex. 10. Three cubics go through seven points. Show that the lines joining the remaining intersections of the cubics taken in pairs form a triangle whose vertices lie one on each of the three given cubics.

[Denote the seven points by P, and the intersections of cubics 2 and 3, 1 and 3 by A, B respectively. Let the lines joining the points A, B meet cubic 3 in H, K. Then on cubic 3 we have

$$P + A \equiv 0, \quad P + B \equiv 0, \quad A + H \equiv 0, \quad B + K \equiv 0.$$

These give $H \equiv K$, so that by Ex. 1 H and K coincide.]

Ex. 11. An n-ic has δ nodes and no cusp. Show that the remaining intersections with the n-ic of a p-ic through the inflexions lie on a $(p - 3n + 2\delta + 6)$-ic which passes through the intersections of the n-ic with its nodal tangents other than the nodes.

[Suppose the n-ic and p-ic meet at the inflexions I and other points P, while the nodal tangents meet the n-ic at points Q and R, of which R coincide with the nodes. Then $P+I \equiv 0$, $Q+R \equiv 0$, $I+R \equiv 0$ (since the points I and R lie on the Hessian), so that $P+Q \equiv 0$.]

Ex. 12. An n-ic has a single node O. Show that the remaining intersections with the n-ic of a p-ic through the points of contact of the tangents from O and the intersections of the n-ic with the tangents at O (other than O) lie on a $(p-n+3)$-ic.

[As in Ex. 11, letting I be the points of contact of tangents from O, and replacing the Hessian by the first polar of O.]

Ex. 13. The sixteen inflexions of a quartic with a biflecnode lie on another quartic, and the eight inflexions of a quartic with two biflecnodes lie on a conic.

[Denote by I the inflexions and by R the intersections of the 4-ic with the nodal tangents. Then from the Hessian $I+R \equiv 0$, and from the nodal tangents $R \equiv 0$, so that $I \equiv 0$. See Ch XVIII, § 1, Ex. 6; § 9, Ex. 6.]

§ 10.

In § 9 we made use of the theorem that, if the curve $f(x, y) = 0$ passes through the intersections of $\phi(x, y) = 0$ and $\psi(x, y) = 0$, then we can put f in the form $A\phi + B\psi$, where A and B are polynomials in x and y.

We are here using Cartesian coordinates, and taking f, ϕ, ψ as the $-C'_l C'_\lambda$, C'_m, C'_n of § 9.

We shall suppose that the intersections of $\phi = 0$, $\psi = 0$ are ordinary distinct points of $f = 0$, $\phi = 0$, $\psi = 0$, and give a proof of the result with this limitation. We may suppose the axes of reference taken perfectly generally.

Let (a, b) be an intersection of $\phi = 0$ and $\psi = 0$. To eliminate y between $\phi = 0$ and $\psi = 0$ we may carry out the process of finding the highest common factor of ϕ and ψ, considered as polynomials arranged in descending powers of y, till we reach a remainder R involving x but not y. Equating this remainder to zero we get the result of elimination required, which is an equation giving the abscissae of the intersections of $\phi = 0$ and $\psi = 0$. The remainder R must have therefore $x-a$, but not $(x-a)^2$, as a factor. Moreover, it follows at once from the process of finding the highest common factor that each remainder found in the process, and in particular the last remainder R, is of the form $\lambda\phi + \mu\psi$, where λ and μ are polynomials in x and y.

Suppose that, when λf is divided by ψ, the quotient is ν and the remainder θ, so that

$$\lambda f = \nu\psi + \theta,$$

ν and θ being polynomials in x and y.

It only remains to show that θ has R as a factor. For, if that is proved, we have $\theta = AR$ or $\theta = A(\lambda\phi + \mu\psi)$, A being a polynomial.

We should then have

$$\lambda f = \nu\psi + A(\lambda\phi + \mu\psi)$$

or $\lambda(f - A\phi) = (\nu + A\mu)\psi$.

Now, since $R = \lambda\phi + \mu\psi$ and R is independent of y, any common factor of λ and ψ would be independent of y. But, if we have chosen the axes of reference generally, ψ has not a function of x as factor. Hence λ and ψ have no common factor, and therefore $(\nu + A\mu)/\lambda$ must be a polynomial B; or $f = A\phi + B\psi$, as required.

To show that θ has R as a factor, we prove that θ has $x - a$ as a factor. A similar argument will then apply to each factor of R, and the result follows.

Suppose ψ of degree n in x and y, and therefore θ of degree $n-1$. The relation $R = \lambda\phi + \mu\psi$ shows that every one of the n intersections of $\psi = 0$ with $x = a$ lies on $\lambda = 0$ or $\phi = 0$, since R has $x - a$ as a factor. But (the axes of reference being general) the only intersection of $\phi = 0$ and $\psi = 0$ with abscissa a is (a, b). Hence the $n-1$ intersections of $\psi = 0$ and $x = a$ other than (a, b) lie on $\lambda = 0$.

Then $\theta = \lambda f - \nu\psi$ shows that all the n intersections of $\psi = 0$ and $x = a$ lie on $\theta = 0$, for (a, b) lies on $f = 0$ and $\psi = 0$, while the other $n-1$ lie on $\psi = 0$ and $\lambda = 0$. But $\theta = 0$, being of degree $n-1$, cannot meet $x = a$ in more than $n-1$ points unless $x - a$ is a factor of θ.

For the case in which the intersections of $f = 0$, $\phi = 0$, $\psi = 0$ are multiple points of any of these curves the reader may refer to Noether's paper in the *Mathematische Annalen* xl (1892), p. 140,* from which the above proof has been adapted. We have applied the residuation theorem in the examples of § 9 to such cases, which was not strictly lawful without a more careful investigation. But the results there given can also be established in most cases by other methods.

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* The reader may consult the bibliography in the *Mathematical Encyclopaedia*.

CHAPTER XIII

UNICURSAL CUBICS

§ 1. Types of Cubic.

SINCE a line joining two double points of a curve meets it in at least four points, a cubic cannot have more than one double point. Hence there are only three types of cubic,* whose Plücker's numbers are given by the table

Type	n	m	δ	κ	τ	ι	D
(i)	3	6	0	0	0	9	1
(ii)	3	4	1	0	0	3	0
(iii)	3	3	0	1	0	1	0

In this chapter we shall discuss the types (ii) and (iii) whose deficiency is zero. These are the nodal and cuspidal cubics. They are unicursal, and the coordinates of any point on the curve may be expressed rationally in terms of a parameter by considering the intersection of the curve with any straight line through the double point.

§ 2. Geometrical Methods applied to Unicursal Cubics.

The properties of a unicursal cubic may be derived from those of a conic by projection and inversion.

There are lines meeting the cubic in only one real point. For let any tangent meet the curve again at P. The point P must be real, since the tangent meets the curve at two real (coincident) points at its point of contact, and therefore meets the curve in three real points. If through P two lines are drawn adjacent to the tangent, one of them evidently meets the curve in three real points and the other in one (see Fig. 1).

Suppose then that a real line meets the curve in unreal points ω and ω'. Project ω and ω' into the circular points at infinity. Then, if the origin O is taken at the double point of the cubic, the curve is now a circular cubic with an equation of the form

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* By curves of 'the same type' we mean those with the same Plücker's numbers.

$$ax^2+2hxy+by^2+2(gx+fy)(x^2+y^2) = 0.$$

Inverting with respect to a circle with centre O and unit radius, we obtain the conic through O

$$ax^2+2hxy+by^2+2gx+2fy = 0.$$

This conic is a parabola, ellipse, or hyperbola, according as the cubic has a cusp, acnode, or crunode; i.e. according as

$$h^2 =, \ <, \text{ or } > ab.$$

The real asymptote of the cubic inverts into the circle of curvature of the conic at O, and the inflexional tangents of the cubic invert into the other osculating circles of the cubic which pass through O.

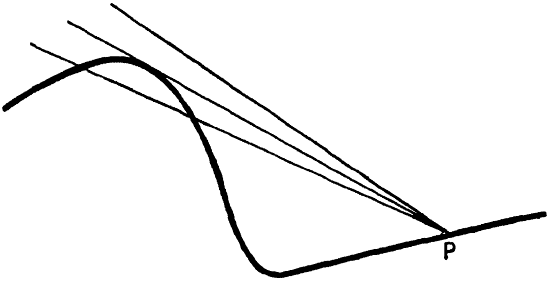

Fig. 1.

Ex. 1. A cubic with an acnode has three real inflexions, and a cubic with a crunode has one real and two unreal inflexions.

[Take the case of the acnodal cubic. By a real projection it may be transformed into a circular acnodal cubic, and then inverting with respect to the acnode we get: 'Through any point O of an ellipse three real circles of curvature pass other that the circle of curvature at O.'

In fact, if the eccentric angle of O is ϕ, the eccentric angles of the three points of contact of the circles of curvature are

$$-\tfrac{1}{3}\phi, \quad \tfrac{1}{3}(2\pi-\phi), \quad \tfrac{1}{3}(4\pi-\phi).$$

Since the sum of these is $2\pi-\phi$, the three points of contact are concyclic with O, and therefore the three inflexions are collinear (cf. Ch. VII, § 4). Similarly for the crunodal cubic.]

Ex 2. The real asymptote of a cuspidal circular cubic is equally inclined to the tangent at the cusp and to the line joining the cusp and the focus.

[Inverting with respect to the cusp and remembering that the focus inverts into a focus (Ch. V, § 4) we have: 'Any tangent to a parabola is equally inclined to the axis and to the focal distances of the point of contact.']

Ex. 3. The lines joining the node of a nodal circular cubic to the real foci are equally inclined to the real asymptote.

Ex. 4. A chord of a circular cubic with a node O subtends 90° at O. Show that the middle point of the chord lies on a fixed straight line, and that the circles on such chords as diameter are coaxial.

Ex. 5. If O is the node of a circular cubic with real foci S, S' and the circle OSS' meets the curve again at A and A',

$$\frac{1}{SO} \pm \frac{1}{S'O} \pm \frac{AA'}{AO \, . \, A'O} = 0.$$

Ex. 6. If O is the centre of a fixed circle touching two given circles externally, find the locus of the inverse of O with respect to any other circle touching the given circles externally.

[Part of a circular cubic with a crunode at O.]

Ex. 7. Obtain properties of a nodal circular cubic by inverting other properties of the conic.

Ex. 8. Any line meets the cissoid $x(x^2+y^2) = ay^2$ with cusp O in P_1, P_2, P_3; and any circle meets it in Q_1, Q_2, Q_3, Q_4. Show that the sum of the cotangents of the angles which

$$OP_1,\ OP_2,\ OP_3 \quad \text{or} \quad OQ_1,\ OQ_2,\ OQ_3,\ OQ_4$$

make with the tangent at the cusp is zero.

[Invert with respect to O. The cissoid becomes a parabola with vertex O.]

Ex. 9. A cubic has three asymptotes and a node O. If a line through O meets the cubic in Q and the asymptotes in P_1, P_2, P_3, show that

$$OP_1 + OP_2 + OP_3 = OQ.$$

Ex. 10. The chord of a circular cubic with a node O subtending 90° at O envelops a conic.

[Expressing the condition that two of the lines joining the intersections of

$$\lambda x + \mu y + 1 = 0 \quad \text{and} \quad x(x^2+y^2) + ax^2 + 2hxy + by^2 = 0$$

to the origin are perpendicular, we get the tangential equation of the envelope.]

Ex. 11. The locus of the point of contact of a tangent from O to a family of confocal conics is a circular cubic through the foci with a node at O.

Consider the case in which O is on an axis or at infinity.

Consider the case of confocal parabolas.

Generalize by projection.

Ex. 12. The locus of the centre of a circle whose circumference passes through two given points and meets a given line not coplanar with the points is a nodal circular cubic.

Ex. 13. A ray of light proceeds from a fixed point A, is reflected at P from any sphere with a given centre O, and after reflexion passes through a fixed point B. Find the locus of P.

[$(x^2+y^2)\{y(b+a)\cos\alpha + x(b-a)\sin\alpha\} = 2abxy$, $z = 0$; if O, A, B are the origin, $(a\cos\alpha, a\sin\alpha, 0)$, $(b\cos\alpha, -b\sin\alpha, 0)$.]

§ 3. Cuspidal Cubics.

A real cuspidal cubic has one cusp and one inflexion, and these must be real. Take the triangle of reference ABC so that B is the inflexion and C the cusp, while AB and AC are the tangents at B and C (Fig. 2).* Then there is no term in the equation of the cubic involving z^3 or z^2; and the only term involving z is zy^2, since $y^2 = 0$ are the tangents at C.

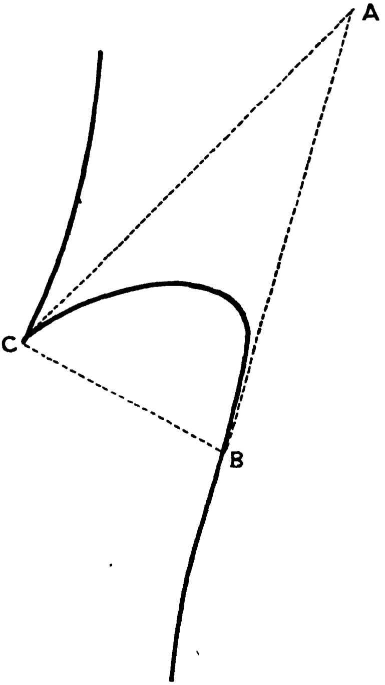

Fig. 2.

Also, when we put $z = 0$, the equation must reduce to $x^3 = 0$, since $z = 0$ meets the curve three times at B. Hence the equation of the cubic is $zy^2 = ax^3$. Putting az for z, we see that

A cuspidal cubic can be put into the form $zy^2 = x^3$ *by a real choice of homogeneous coordinates.*

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* The curves shown in Figs. 2, 3, 4 have the Cartesian equations $2y^2(x-1)-2y(x^2-2x)+x^2(x-2)=0$, $64y^2(x-4)+16x^2y+x^2(5x-12)=0$, and $x(x^2-xy+4y^2)+4y(2x-y)=0$ respectively.

It follows from Ch. I, § 3 that any cuspidal cubic can be projected into the semicubical parabola $ay^2 = x^3$.

Any point on the curve $zy^2 = x^3$ may be taken as $(t, 1, t^3)$, t being the parameter of the point.

If the points with parameters t_1, t_2, t_3 lie on the line

$$\lambda x + \mu y + \nu z = 0,$$

t_1, t_2, t_3 are the roots of the equation in t

$$\lambda t + \mu + \nu t^3 = 0,$$

so that $t_1 + t_2 + t_3 = 0$.

Hence:

If three points of $zy^2 = x^3$ are collinear, the sum of their parameters is zero, and conversely

If the tangent to any cubic (unicursal or otherwise) at P meets the curve again at Q, Q is called the *first tangential*, or simply 'the tangential' of P. The tangential of the first tangential is called the *second tangential*, the tangential of the second tangential is called the *third tangential*, and so on.

If the parameter of P is t, the parameter of the first tangential Q is $-2t$, for the tangent at P meets the curve in the points P, P, Q whose parameters have a zero sum.

The parameter of the n-th tangential of P is evidently $(-2)^n t$.

If the points with parameters $t_1, t_2, t_3, t_4, t_5, t_6$ lie on the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

these quantities are the roots of

$$at^2 + b + ct^6 + 2ft^3 + 2gt^4 + 2ht = 0.$$

Since the coefficient of t^5 in this equation is zero, we have:

If six points of $zy^2 = x^3$ lie on a conic, the sum of their parameters is zero, and conversely.

Similarly:

If nine points of $zy^2 = x^3$ are the intersections of this cubic with another cubic, the sum of their parameters is zero, and conversely.

Ex. 1. Any cuspidal cubic can be projected into

$$a^2 y = x^3, \quad \text{or into} \quad yx^2 = a^3.$$

Ex. 2. The successive tangentials of a given point approach the cusp as their limiting position.

[$\mathop{\mathrm{Lt}}\limits_{n\to\infty} (-2)^n t = \infty$, and the parameters of the inflexion and cusp are zero and infinite.]

Ex. 3. From a point P on a cuspidal cubic a tangent is drawn touching at P_1, from P_1 a tangent is drawn touching at P_2, and so on. Show that P_n approaches the inflexion as its limiting position.

[The parameters of P and P_n are in the ratio $1 : (-\frac{1}{2})^n$.]

Ex. 4. The conic of closest contact at P to a cuspidal cubic meets it again in P_1, similarly P_2 is derived from P_1, and so on. Find the limiting position of P_n.

[The parameters of P and P_1 are in the ratio $1 : -5$.]

Ex. 5. Through a point P of a cuspidal cubic a conic is drawn having 5-point contact at P', similarly P'' is derived from P', and so on. Find the limiting position of $P^{(n)}$.

Ex. 6. A conic passes through two given points of a cuspidal cubic, osculates it at P, and cuts it again at P_1. Similarly P_2 is derived from P_1, P_3 from P_2, and so on. Find the limiting position of P_n.

Ex. 7. The conic of closest contact at any point of a cuspidal cubic (other than the cusp or inflexion) meets the inflexional tangent in unreal points and the cuspidal tangent in real points.

[The conic of closest contact at $(t, 1, t^3)$ with $zy^2 = x^3$ is

$$45\,t^4x^2+5\,t^6y^2-z^2-40\,t^3yz+15\,t^2zx-24\,t^5xy = 0,$$

as is seen by putting $\theta, 1, \theta^3$ for x, y, z in the general equation of a conic and comparing the resulting equation for θ with

$$(\theta-t)^5(\theta-t') = 0.$$

See also Ch. X, § 2, Ex. 16 (i).]

Ex. 8. The loci of the poles of the inflexional and cuspidal tangents with respect to the conic of closest contact at any point of a cuspidal cubic are cuspidal cubics with the same cusp, inflexion, cuspidal tangent, and inflexional tangent.

Ex. 9. A conic osculates a cuspidal cubic at P and Q. Show that the line PQ passes through the inflexion and is divided harmonically by the inflexion and the tangent at the cusp.

Ex. 10. Find the locus of the intersection of the tangent at P to a cuspidal cubic with the tangent at the second tangential of P.

Ex. 11. The tangent at P to a cuspidal cubic meets the curve, the inflexional tangent, the cuspidal tangent in Q, R, T respectively. Show that the cross-ratio of the range $(PQRT)$ is 9/8.

Ex. 12. If in Ex. 11 the cross-ratio of $(PQRS)$ is any constant other than 9/8, find the locus of S.

[A cubic with the same cusp, inflexion, cuspidal tangent, and inflexional tangent.]

Ex. 13. The tangents from P to a cuspidal cubic meet the curve again in three points. Show that the tangents at these points meet in a point Q; and that, if P moves along a straight line, so does Q.

Ex. 14. Obtain theorems from Ex. 7 to 13 by reciprocation.

Ex. 15. Find the locus of the intersection of tangents to the semicubical parabola $ay^2 = x^3$ at points subtending 90° at the cusp.

[A parabola having double contact with the given curve.]

Ex. 16. The locus of the intersection of two perpendicular normals to a semicubical parabola is a nodal cubic.

Ex. 17. The locus of a point P, which moves so that the sum of the angles made with a fixed line by the tangents from P to a semicubical parabola is constant, is a straight line.

Ex. 18. The chord PQ of the cissoid $x(x^2+y^2) = ay^2$ subtends 90° at the cusp. Show that

(i) The locus of the middle point of PQ is a straight line.

(ii) The locus of the intersection of the tangents at P and Q is a circle.

(iii) The locus of the intersection of the normals at P and Q is a straight line.

[Any point on the curve is $x = a/(1+t^2)$, $y = a/t(1+t^2)$. See § 2, Ex. 8, and Ch. XI, § 8, Ex. 10.]

Ex. 19. Find the envelope of the common chord of a cissoid and its circle of curvature.

[If four points on the curve are concyclic, their parameters have zero sum. The chord joining the points with parameters t, $-3t$ is

$$(1+7t^2)x = 6t^3y + a \quad \text{enveloping} \quad 3^5y^2(a-x) = 7^3x^3.]$$

Ex 20. The locus of the focus of a parabola passing through fixed points P and Q, and having PQ as normal at P, is the cissoid of Ex. 18.

[If the parabola is $y^2 = 4ax$ and P is $(at^2, 2at)$, while SZ is the perpendicular from the focus S on PQ,

$$SZ = at(1+t^2)^{\frac{1}{2}}, \qquad PZ = a(1+t^2)^{\frac{1}{2}}, \qquad PQ \,.\, t^2 = 4a(1+t^2)^{\frac{3}{2}}.$$

Now eliminate a and t.]

Ex. 21. The sum of the abscissae of the feet of concurrent normals of $a(x-y)^2 = x^3$ is constant.

Ex. 22. The equation of any n-ic with a tangent of n-point contact and a superlinear branch of order $n-1$ may be put in the form

$$zy^{n-1} = x^n.$$

§ 4. Nodal Cubics.

The equation of any acnodal cubic can be put in the form

$$z(x^2+y^2) = y(3x^2-y^2)$$

by a suitable choice of homogeneous coordinates.

A real nodal cubic has a real node and three inflexions, of which one at least must be real. Take the triangle of reference ABC so that C is the node, A is a real inflexion, and CB is the harmonic conjugate of CA with respect to the tangents at C, i.e. CB is the harmonic polar of the inflexion A.

Since C is a node, and the tangents at C form a harmonic pencil with $xy = 0$, the coefficients of z^3, xz^2, yz^2, xyz in the equation of the cubic are zero. Since A is an inflexion, the coefficients of x^3 and xy^2 must be zero, remembering that the tangent at A meets the curve thrice at A.

Moreover, since C is an acnode, the coefficients of x^2z and y^2z have the same sign. Choosing suitable homogeneous coordinates, we may make these coefficients the same, and the equation takes the form

$$z(x^2+y^2)+y(ax^2+by^2)=0.$$

In this replace z by $\frac{1}{4}(b-a)z-\frac{1}{4}(3b+a)y$, which by Ch. I, § 1 is equivalent to choosing the line

$$z+\tfrac{1}{4}(3b+a)y=0$$

as the side AB of a new triangle of reference, and the equation reduces to

$$z(x^2+y^2)=y(3x^2-y^2) \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

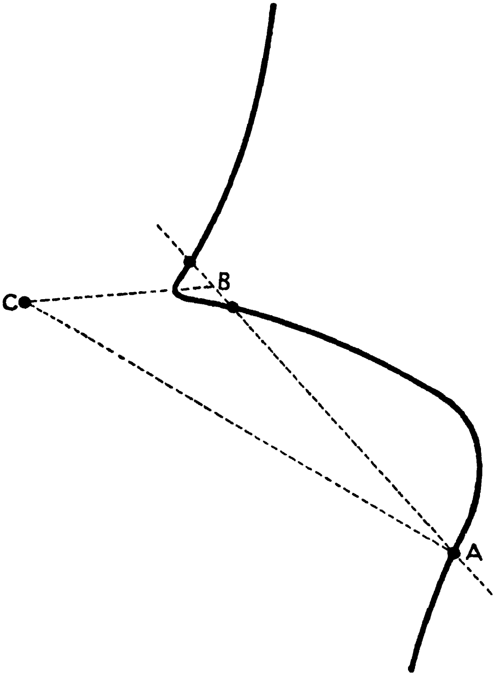

Fig. 3.

The Hessian of (i) is

$$z(x^2+y^2)=-3y(3x^2-y^2) \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

It meets the curve (i) where $z=0$. Hence the line $z=0$ passes through the three inflexions $(1, 0, 0)$, $(1, \pm\sqrt{3}, 0)$ of the cubic (i).

Any point on the cubic (i) is

$$(\cos\phi, \quad \sin\phi, \quad \sin 3\phi).$$

The whole curve, except the acnode, is obtained by giving ϕ all real values between 0 and π (Fig. 3).

If the points with parameters ϕ_1, ϕ_2, ϕ_3 lie on the line

$$\lambda x + \mu y + \nu z = 0,$$

$\tan\phi_1, \tan\phi_2, \tan\phi_3$ are the roots of

$$(\cos^2\phi + \sin^2\phi)(\lambda\cos\phi + \mu\sin\phi) + \nu\sin\phi\,(3\cos^2\phi - \sin^2\phi) = 0,$$

considered as an equation in $\tan\phi$. Therefore

$$\tan\phi_1 + \tan\phi_2 + \tan\phi_3 = \tan\phi_1\ \tan\phi_2\ \tan\phi_3.$$

Hence:

If three points on the cubic (i) *with parameters* ϕ_1, ϕ_2, ϕ_3 *are collinear,* $\phi_1 + \phi_2 + \phi_3 \equiv 0$ *(mod.* π*); and conversely.*

Putting ϕ_1, ϕ_2, ϕ_3 all equal, we obtain the parameters of the inflexions, namely $0, \frac{1}{3}\pi, \frac{2}{3}\pi$. The sum of these is π. Hence:

An acnodal cubic has three real collinear inflexions;

as found before by means of the Hessian.

Similarly we may show that:

If six points on the cubic (i) *with parameters* $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$ *lie on a conic,* $\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 \equiv 0$ *(mod.* π*); and conversely.*

The equation of any crunodal cubic can be put in the form

$$z(x^2 - y^2) = y(3x^2 + y^2)$$

by a suitable choice of homogeneous coordinates.

The proof is almost exactly the same as that given for the acnodal cubic, except that the coefficients of x^2z and y^2z have opposite signs.

The Hessian of

$$z(x^2 - y^2) = y(3x^2 + y^2) \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

is

$$z(x^2 - y^2) = -3y(3x^2 + y^2) \quad . \quad . \quad . \quad . \quad \text{(iv)}.$$

As before, $z = 0$ is the line of inflexions.

The coordinates of any point on (iii) are

$$(\cosh\phi, \quad \sinh\phi, \quad \sinh 3\phi).$$

The condition for three points on a line or six points on a conic is the same as that given for the acnodal cubic, substituting (mod. πi) for (mod. π).

The inflexions have parameters $0, \frac{1}{3}\pi i, \frac{2}{3}\pi i$.

Hence:

A crunodal cubic has three collinear inflexions of which only one is real.

If we suppose that the points of (iii) at the two ends of an asymptote are the same, we may say that the whole curve is divided into two portions by the crunode. One contains the real inflexion, and is obtained by giving ϕ all real values from $-\infty$ to $+\infty$. The other portion, which we shall call the *loop*, is obtained by giving ϕ the value $\theta+\frac{1}{2}\pi i$, where θ takes all real values between $-\infty$ and $+\infty$ (Fig. 4).

The coordinates of any point on a nodal cubic can be expressed *rationally* in terms of a parameter t, by putting $\cot\phi = t$ or $\coth\phi = t$.

Many convenient standard forms of the equation of a nodal cubic exist, other than (i) and (iii).

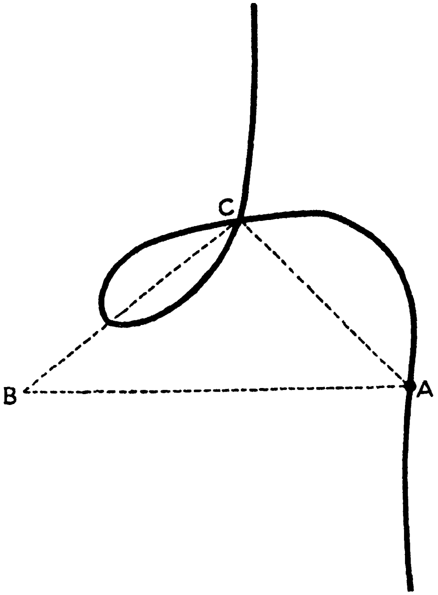

Fig. 4.

We may mention the form

$$x(z^2 \pm y^2) = y^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

obtained by taking A as the node, C as an inflexion, AB as the harmonic polar of C, and CB as the tangent at C. Any point on the curve is

$$(\sin^3\phi,\ \sin\phi,\ \cos\phi) \text{ or } (\sinh^3\phi,\ \sinh\phi,\ \cosh\phi).$$

The condition for three points on a line or six points on a conic is the same as that given above. The line of inflexions is $4x = \pm 3y$.

The form (v) is suggested by that given in Ch. XVI, § 1 for the non-singular cubic.

Ex. 1. A nodal cubic can be projected into

$$a(x^2 \pm y^2) = x^3 \quad \text{or into} \quad y(x^2 \pm a^2) = a^3.$$

Ex. 2. Show from the form $y(x^2 \pm a^2) = a^3$ that a nodal cubic has three or one real inflexions according as it has an acnode or crunode.

[$\frac{d^2y}{dx^2} = 2a^3(3x^2 \mp a^2)/(x^2 \pm a^2)^3$ vanishes at an inflexion.]

Ex. 3. Show that the tangents at two inflexions of a nodal cubic meet on the harmonic polar of the third inflexion.

[The tangents at the finite inflexions of $y(x^2+a^2) = a^3$ meet on $x = 0$, which is the harmonic polar of the inflexion $(\infty, 0)$.]

Ex. 4. Show that the equation of any crunodal cubic can be put in the form $x^3+y^3 = 3xyz$.

Show that, if any point on the curve is taken as $(3t, 3t^2, 1+t^3)$, the product of the parameters of three collinear points is -1 and the product of the parameters of six points on a conic is $+1$.

Show that any crunodal cubic can be projected into the folium of Descartes $x^3+y^3 = 3axy$.

Ex. 5. A line meets a nodal cubic in P, Q, R, and the lines joining these points to the node make angles θ_1 and θ_2, ϕ_1 and ϕ_2, ψ_1 and ψ_2 with the tangents at the node. Show that

$$(\sin\theta_1 \,.\, \sin\phi_1 \,.\, \sin\psi_1) \div (\sin\theta_2 \,.\, \sin\phi_2 \,.\, \sin\psi_2)$$

is constant.

[Use the fact that in Ex. 4 the product of the parameters of three collinear points is -1.]

Ex. 6. Show that no real tangent can be drawn to a crunodal cubic from a point on the loop; and that from any other point of the curve two real tangents can be drawn, one point of contact being on the loop and the other not.

[If ϕ, ψ are the parameters of a point and its tangential,

$$2\phi + \psi \equiv 0 \pmod{\pi i}.]$$

Ex. 7. Considering an acnodal cubic (supposed continuous at the two ends of an asymptote) as divided into three parts by the three real inflexions, prove that from any point on one part two real tangents can be drawn to the curve, one point of contact lying on each of the other two parts.

Ex. 8. The conic of closest contact at P with a crunodal cubic meets the line of inflexions at unreal or real points according as P does or does not lie on the loop.

[See Ex. 4 and Ch. X, § 2, Ex. 16 (ii).]

Ex. 9. The line of inflexions of

$$(a_0x^2 + 2a_1xy + a_2y^2)z + A_0x^3 + 3A_1x^2y + 3A_2xy^2 + A_3y^3 = 0$$

is $4(a_0a_2 - a_1^2)z + 3(a_0A_2 + a_2A_0 - 2a_1A_1)x + 3(a_0A_3 + a_2A_1 - 2a_1A_2)y = 0$.

[It is the $(n-1)a_{nn}z = -(n-2)a_{nr}$ of Ch. VII, § 7, Ex. 14 (v) in the case $n = 3$. For another proof see Bromwich, *Messenger Math.*, xxxii, p. 113. Or again, we may use the theory of covariants, verifying the result for any special form of the equation of the cubic.]

Ex. 10. A family of nodal cubics has given inflexions and the tangents at two of the inflexions are also given. Show that the locus of the node is a straight line.

[The harmonic polar of the third inflexion by Ex. 3. Or put the equation of the family in the form

$$k^3z^3 = (x+y+3kz)\,xy.]$$

Ex. 11. The line joining two points of the cubic of § 4 (i) with parameters α, β is

$$z = \{\sin 2\alpha + \sin 2\beta - \sin 2(\alpha+\beta)\}\,x + \{\cos 2\alpha + \cos 2\beta + \cos 2(\alpha+\beta)\}\,y.$$

For the cubic of § 4 (iii) replace sin and cos by sinh and cosh.

Ex. 12. The other intersections with the curve of any line through a fixed point of a nodal cubic subtend an involution at the node.

Ex. 13. The tangents at Q and R to a cubic with node O meet on the curve at P. The line QR meets the curve again at S, and OP meets QR at E and the line of inflexions at F. Show that

(i) OQ and OR are harmonically conjugate with respect to the tangents at O and also with respect to OP and OS.

(ii) (OF, EP) is harmonic.

(iii) As P varies QR envelops a conic touching the tangents at the node, the line of inflexions being the chord of contact.

[Take the cubic of § 4 (i); the case of the cubic (iii) is similar. The parameters of P, Q, R, S are respectively ϕ, $-\frac{1}{2}\phi$, $\frac{1}{2}(\pi-\phi)$, $\phi-\frac{1}{2}\pi$. The line QR is $z + \sin 2\phi\, x + \cos 2\phi\, y = 0$, whose envelope is $z^2 = x^2 + y^2$.]

Ex. 14. The tangents at Q and R to a cubic with a node O meet on the curve at P, and the tangents at Q' and R' meet on the curve at P'. Show that

(i) The conic $OQRQ'R'$ meets the line of inflexions on OP and OP'.

(ii) The conic through O touching the cubic at Q and R touches the line of inflexions at a point on OP.

(iii) If the tangents at P and P' meet on the curve, QR and $Q'R'$ meet on the line of inflexions.

Ex. 15. A nodal cubic and its Hessian are in plane perspective, the node being the vertex of perspective, and the line of inflexions the axis of perspective.

Ex. 16. The points of contact of the four tangents from P to a nodal cubic subtend a pencil of constant cross-ratio at the node. Find the locus of P.

[Eliminating z between the equation of the cubic and the polar conic of P, we get the rays of the pencil. Now use Ch. I, § 11.

If the pencil is equianharmonic, the locus is the line of inflexions. If it is harmonic, the locus is a cubic.]

Ex. 17. A conic touches the tangents at the node of a cubic. Prove that the points of contact with the cubic of the six other common tangents of conic and cubic lie on another conic.

Ex. 18. A straight line cuts $a(x^2+y^2)=x^3$ in P, Q, R. Show that the lines joining P, Q, R to the node make angles with any fixed line whose sum is constant.

[Any point on the curve is $(a(1+t^2), at(1+t^2))$. If t_1, t_2, t_3 are the parameters of three collinear points, $t_2t_3+t_3t_1+t_1t_2=1$, which shows that the lines joining the points to the origin make angles with $x=0$ whose sum is $\frac{1}{2}\pi$.]

Ex. 19. Prove a similar theorem for the six points in which the cubic of Ex. 18 meets any conic.

Ex. 20. Show that the points of contact of tangents from any point on $3x=2a$ to $a(x^2-y^2)=x^3$ lie on a circle through the origin, and that the centre of this circle lies on a fixed line.

Ex. 21. A circle through the node cuts $a(x^2-y^2)=x^3$ again in four points, and the sum of the angles which the lines from the node to these points makes with a fixed line is constant. Show that the locus of the centre of the circle is a fixed line.

Ex. 22. If a chord of $x^3+y^3=3axy$ subtends 90° at the node, the chord meets the cubic again at a fixed point.

CHAPTER XIV

NON-SINGULAR CUBICS

§ 1. Cubics with Unit Deficiency.

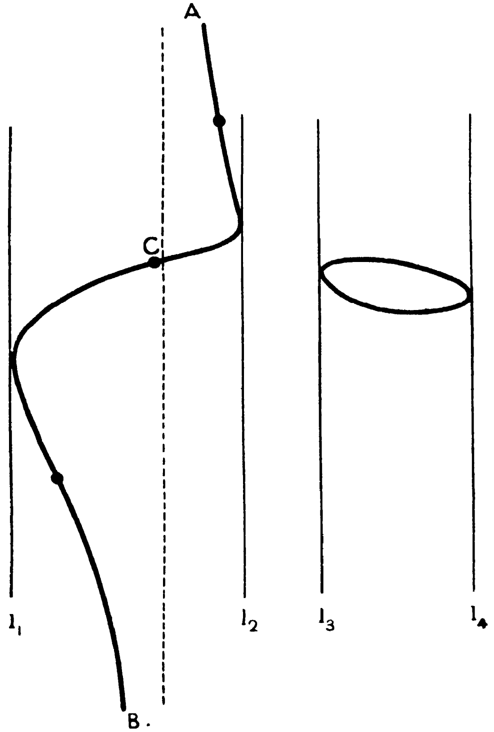

Fig. 1.
$9(x+4)y^2+24xy+x(x^2+2x-16) = 0.$

In this chapter and the two following we shall consider the properties of non-singular cubics, i.e. cubics without a double point and of unit deficiency.

As in Ch. XIII, § 2, we see that there are lines meeting the curve in only one real point. Project such a line to infinity, and the curve will have one and only one real asymptote.

Place the asymptote vertical (Fig. 1), and suppose a line l starts from coincidence with the asymptote and travels to the right, remaining parallel to the asymptote.

At first l meets the curve in two real finite points, for the curve in general approaches the asymptote on opposite sides at the two ends. As l travels, eventually these two points coincide and l is a tangent. Similarly if l travels to the left. Hence there is a continuous portion of the cubic called the *odd circuit* * approaching the asymptote at its ends, meeting the asymptote in a finite point, and contained between two lines l_1, l_2 parallel to the asymptote.

It may happen that, when l travels still further to the right (or left), it touches the cubic again. As l travels yet further it must again become a tangent, since the curve has only one asymptote. The curve will then have a portion called the *even circuit* * or *oval* contained between two lines l_3, l_4 parallel to the asymptote. The curve can have no other portion; for, if two ovals existed, a straight line cutting both would meet the curve in four points.

If a cubic has three real asymptotes, the term 'even circuit' is applied to that part of the curve (if any) which can be projected into an oval, the rest of the curve being the 'odd circuit'.

Hence cubics can be classified into two main divisions: *one-circuited cubics* having a single odd circuit, and *two-circuited cubics* having an odd and an even circuit.†

From a point P on the even circuit of a cubic, no real tangent can be drawn (other than the tangent at P). For any line through P must evidently meet the odd circuit and must meet the even circuit in a point other than P (see fig. 1). Of the three points thus obtained no two can coincide, unless the line is the tangent at P.

From a point P on the odd circuit two real tangents can evidently be drawn to the even circuit (if any). Moreover, two real tangents can be drawn from P to the odd circuit. For this is true of one point of the circuit, namely the distant point A; and as a point travels along the curve from A to P, a real tangent from the point remains real and an unreal tangent remains unreal.

The cubic has nine inflexions, namely the nine intersections of the cubic with its Hessian (Ch. VII, § 7). Of these an odd number must be real, since both cubic and Hessian are real. We shall show that exactly three are real, that they are collinear, and that they lie on the odd circuit of the cubic.

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* The names 'odd' or 'even' circuit emphasize the fact that the circuit is met by any straight line respectively in an odd or even number of points.

† The names 'unipartite' and 'bipartite' are also used instead of 'one-circuited' and 'two-circuited'.

Firstly, no inflexion can lie on the even circuit. For the tangent at such an inflexion would meet the cubic at three points coinciding with the point of contact. But every straight line must evidently meet the odd circuit in at least one point, and therefore no straight line can meet the even circuit in more than two points.

Secondly, suppose A and B very distant points on opposite ends of the odd circuit. It is evident from fig. 1 that, since the odd circuit does not cross itself * between A and its intersection C with the asymptote, the circuit has at least one inflexion between A and C. Similarly it has an inflexion between B and C.

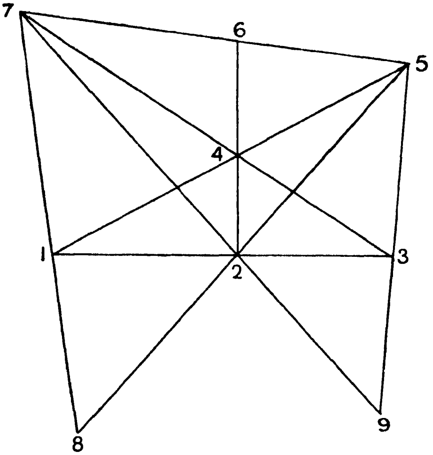

FIG. 2.

The circuit has then two real inflexions. The line joining them meets the circuit again in a real point which is also an inflexion (Ch. VII, § 4).

To show that the curve has no more than three real inflexions, suppose that I_1, I_2, I_3 are the inflexions just found and that I_4 is a fourth real inflexion (Fig. 2). Then I_1I_4, I_2I_4, I_3I_4 meet the curve again in real inflexions I_5, I_6, I_7; while I_1I_7 and I_3I_5 meet the curve again in real inflexions I_8 and I_9, which we may suppose to be their intersections with I_2I_5 and I_2I_7, since these latter lines also meet the curve again in inflexions. Then I_6I_1, I_6I_3, I_6I_8, I_6I_9, &c., meet the curve

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* Contrast the case of the crunodal cubic in Ch. XIII, fig. 4.

again in real inflexions; contrary to the fact that the curve has no more than nine inflexions real or unreal.*

Ex. 1. Obtain the results of § 1 by proving that at least one line can be drawn through an inflexion meeting the curve in only one real point, and projecting that line to infinity.

Ex. 2. Obtain the results of § 1 by projecting a real inflexional tangent to infinity.

Ex. 3. Show that the locus of the centroid of the three intersections with a given cubic of straight lines drawn in a fixed direction is a straight line; and that, as the direction varies, the locus envelops a conic.

Ex. 4. Through a point O of a cubic any line is drawn meeting the cubic in P and Q. Show that the locus of the middle point of PQ is a cubic with a node at O and asymptotes parallel to those of the given cubic.

[Use polar coordinates with O as pole; or see Ch. XI, § 8, Ex. 3.]

Ex. 5. The locus of a point P, such that the lines joining P to three fixed points meet three fixed lines in collinear points, is a cubic through the fixed points and the vertices of the triangle formed by the fixed lines.

Ex. 6. If OA, OB, OC, OD are the tangents from a point O of a cubic to the curve, the cubic is the locus of the point of contact of a tangent from O to any conic through A, B, C, D.

§ 2. Circular Cubics.

We pointed out in § 1 that we can always find a line meeting a real cubic in only one real point. We may project the unreal intersections of the line and cubic by a real projection into the circular points ω, ω'. Hence we can always project a cubic into a *circular cubic*, i. e. a cubic passing through the circular points ω, ω'.

From the real point at infinity on a circular cubic two or four real tangents can be drawn according as the cubic has one or two circuits (§ 1). They are parallel to the real asymptote.

Take the point of contact of one of these tangents as the origin, and the tangent as the axis of y. Then the cubic takes the form

$$x(x^2+y^2)+ax^2+2hxy+by^2+kx=0 \quad . \quad . \quad . \quad \text{(i)}.$$

The inverse of this with respect to the circle with centre the origin and radius $k^{\frac{1}{2}}$ is obtained by writing $kx/(x^2+y^2)$ for x, $ky/(x^2+y^2)$ for y, and is found to be the cubic itself. Hence:

---

* We may apply the above proof to show that an acnodal cubic has exactly three inflexions which are real and collinear.

A circular cubic is self-inverse with respect to four circles.*

This system of circles is obviously self-inverse with respect to any one of the circles, since the cubic is. Therefore the circles are all mutually orthogonal. It follows that the radical axis of any pair of the circles must pass through the centres of the other two, and thence that any one centre is the orthocentre of the triangle formed by the other three.

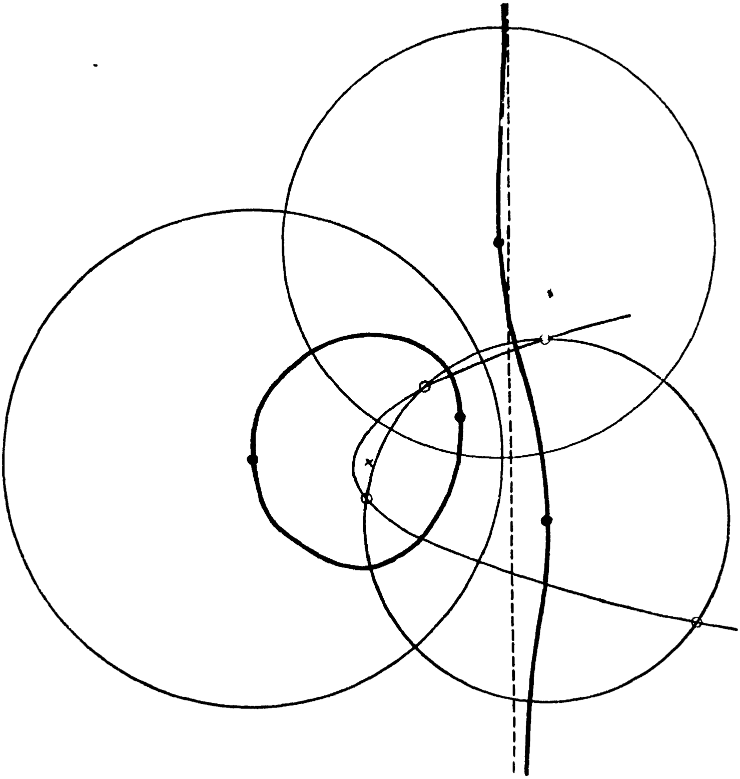

Fig. 3 (i).

If the circular cubic has two circuits, three of the circles with respect to which it is self-inverse are real. The remaining circle is unreal, but it has a real centre on the side of the oval near to the odd circuit.

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* The word *anallagmatic* is sometimes used in the sense of 'its own inverse' or 'self-inverse'.

If the circular cubic has one circuit, two of the circles are real, and the other two have unreal centres. This will be clear from Fig. 3.

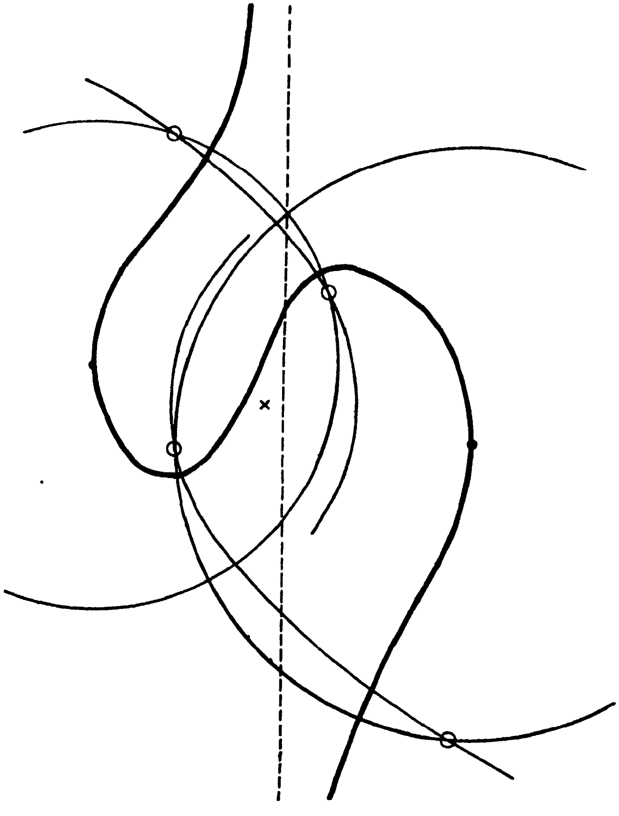

Fig. 3 (ii).

$$4x(x^2+y^2)+4x(x+5y)+40y-71x=0.$$

Ex. 1. The two circles with respect to which a circular cubic is self-inverse and which have their centres on the odd circuit meet the cubic at two or four real points according as the cubic has one or two circuits. The other two circles do not meet the cubic in real points.

[The points are the points of contact of tangents to the cubic from the centres of the circles.]

Ex. 2. The tangents to a circular cubic from a point where the tangent is parallel to the real asymptote are all equal.

Ex. 3. The inverse of a circular cubic with respect to a point on it is a circular cubic. Its inverse with respect to any other point is a bicircular quartic.

Ex. 4. Any circular cubic can be inverted into a circular cubic symmetrical about a line, or into a bicircular quartic symmetrical about two perpendicular lines.

[The pole of inversion is (i) the intersection of the cubic with a circle for which it is self-inverse, (ii) the intersection of two such circles.]

Ex. 5. A nodal circular cubic is self-inverse with respect to two circles, and a cuspidal circular cubic with respect to one circle.

§ 3. Foci of Circular Cubics.

Suppose j is one of the four circles with respect to which a circular cubic is self-inverse. Let a tangent from the circular point ω to the cubic meet j at S. Then the inverses of the cubic and the line ωS with respect to j are the cubic and $\omega' S$ respectively. Hence $\omega' S$ also touches the cubic, and S is a focus. Now four tangents can be drawn from ω to the cubic, other than the tangent at ω, for the cubic is of class 6. Hence:

Any circular cubic is self-inverse with respect to each of four mutually orthogonal circles, and the sixteen foci lie by fours on these four circles.

If P, Q, R, S are the foci on j, the pencils $\omega(PQRS)$ and $\omega'(PQRS)$ have the same cross-ratio, since ω, ω', P, Q, R, S are all on the circle j. Therefore the pencil of tangents from ω to the cubic has the same cross-ratio as the pencil of tangents from ω'. If we allow the use of unreal projection, any two points of a cubic may be projected into ω and ω', the cubic being thus projected into a circular cubic. Hence we have the important theorem:

The tangents to a cubic from any point of the curve (other than the tangent at the point) form a pencil of constant cross-ratio.

The cross-ratio is real, if the cubic has two circuits; unreal or equal to -1, if the cubic has only one circuit. This is evident on considering the pencil of tangents from a real point at infinity on the cubic (see Fig. 1). For the cross-ratio of a pencil formed by two real and two conjugate unreal lines is unreal, unless the pencil is harmonic; while the cross-ratio of a pencil formed by four real lines is real.

It will be seen in § 5 that, if the cross-ratio is -1, then with the notation of that section, $k^2+6k+6=0$. There exist both one-circuited ($k = -3 + \sqrt{3}$) and two-circuited ($k = -3 - \sqrt{3}$) cubics, for which the cross-ratio of tangents is -1.

If a real focus S of a two-circuited circular cubic lies on a circle j for which the cubic is self-inverse, the other real foci are obtained by inverting S repeatedly with respect to the other circles for which S is self-inverse. These inversions leave j unaltered, and therefore all four real foci lie on j. The cross-ratio of the pencil of tangents from any point of the curve is the cross-ratio of the pencil subtended by the four real foci at ω or at any point of the circle j.

Similarly we show that, if the cubic is one-circuited, two real foci lie on each of the two real circles with respect to which the cubic is self-inverse.*

Take now the origin at the singular focus of the circular cubic and the axis of y parallel to the real asymptote. Then the terms of the highest degree in the equation of the curve are $x(x^2+y^2)$, and the lines $x^2+y^2 = 0$ meet the curve in only one finite point apiece. It follows at once that the equation of the cubic is

$$(x+p)(x^2+y^2)+ax+by+c = 0 \quad . \quad . \quad . \quad . \quad \text{(i)},$$

where $x+p = 0$ is the real asymptote.

Consider the circle, whose centre $(tm^2-t,\ 2mt)$ lies on the parabola with focus at the origin

$$y^2 = 4t(x+t) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)},$$

and which cuts orthogonally the circle

$$x^2+y^2+2(p-2t)x+by/2t-(8t^2-4tp+a) = 0 \quad . \quad \text{(iii)}.$$

Its equation is

$$2tm^2(2t-x-p)-m(4ty+b)+x^2+y^2+a+2t(x-p)+4t^2 = 0 \quad . \quad . \quad . \quad . \quad \text{(iv)}.$$

Its envelope is found by writing down the condition that (iv) should have equal roots in m. It coincides with (i) provided

$$64t^4-64pt^3+16(p^2+a)t^2+8(c-ap)t-b^2 = 0 \quad . \quad \text{(v)}.$$

The circle (iv) is orthogonal to the circle (iii), and therefore it is self-inverse with respect to (iii). Hence its envelope (i) is self-inverse with respect to (iii). If t is any given root t_1 of (v), (iii) is one of the circles with respect to which (i) is self-inverse.

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* As just pointed out, if all four real foci lie on j, the cross-ratio of tangents is real, which is only possible for a one-circuited cubic, if the pencil is harmonic. This case requires further investigation; see § 3, Ex. 1, and § 4, Ex. 3.

The centres of those four circles of the family (iv) which degenerate into line-pairs (circles of zero radius) lie both on (iii) and on the parabola obtained by putting $t = t_1$ in (ii). They are by definition foci of (i).

It will be noticed that the distance of the centre of (iii) from the asymptote $x+p = 0$ is $2t$. Summing up, we have:

A circular cubic may be considered in four ways as the envelope of a circle which is orthogonal to one of the circles j with respect to which the cubic is self-inverse, and whose centre lies on a fixed parabola with its focus at the singular focus of the cubic and with its axis perpendicular to the real asymptote of the cubic. The intersections of the parabola with the circle j are foci of the cubic, and the distance of the centre of j from the asymptote is equal to the semi-latus-rectum of the parabola.

The parabolas are called 'focal parabolas' or 'deferent parabolas' of the cubic.

In Fig. 3 (i) the singular focus of the cubic is (0, 0) and the real ordinary foci are

$$(0, -\cdot 5), \quad (\cdot 75, 1), \quad (2\cdot 31, 1\cdot 6), \quad (4\cdot 16, -2\cdot 1).$$

The points at which the tangents are parallel to the real asymptote $200x = 361$, i.e. the centres of the circles with respect to which the cubic is self-inverse, are the centre $(2\cdot 305, -\cdot 715)$ of the circle through the real foci, and the harmonic points $(\frac{42}{25}, \frac{143}{50})$, $(-\frac{63}{40}, \frac{11}{104})$, $(\frac{6}{5}, \frac{13}{22})$ of the quadrangle whose vertices are the real foci.*

The focal parabola through the real foci is $4y^2 = 4x+1$. The other focal parabolas are at once obtained from the fact that their latera recta are twice the distance from the asymptote of the centres of the circles for which the cubic is self-inverse.

In Fig. 3 (ii) the circles for which the cubic is self-inverse are

$$4x^2+4y^2-40x+28y-99 = 0$$

and

$$4x^2+4y^2+40x+12y-59 = 0,$$

while the corresponding focal parabolas are

$$(2y+5)^2 = 40(x+3) \quad \text{and} \quad (2y+5)^2 = -40(x-2).$$

The asymptote is $x = 0$, and the origin lies on the curve.

In Fig. 3 (i) and (ii) the foci are shown by the small circles o, the singular focus by a cross ×, the points of contact of tangents parallel to the asymptote by the dots . .

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* This follows from the fact that foci invert into foci when the cubic is inverted into itself.

Ex. 1. Let a, b, c, d be tangents from ω to a circular cubic, and let a', b', c', d' be tangents from ω', so that the foci aa', bb', cc', dd' lie on the circle j of § 3. Then the foci

(i) aa', bb', cc', dd', (ii) ba', ab', dc', cd',
(iii) ca', db', ac', bd', (iv) da', cb', bc', ad'

are such that the groups (i), (ii), (iii), (iv) each lie on one of the circles for which the cubic is self-inverse. If aa' and bb' are real, the circle (i) contains four real foci, or else the circles (i) and (ii) contain two real foci apiece. No other circle passes through four foci, unless the pencil of tangents from a point on the curve is harmonic.

[The pencil $(a'b'c'd')$ has the same cross-ratio as $(abcd)$, $(badc)$, $(cdab)$, $(dcba)$. If $(abcd)$ is harmonic, $(a'b'c'd')$ and $(cbad)$ have the same cross-ratio, so that ca', bb', ac', dd' are also concyclic.]

Ex. 2. In Ex. 1 the centres of the circles with respect to which the cubic is self-inverse are the centre of the circle (i) and the intersections of the pairs of lines joining aa', bb', cc', dd'.

Ex. 3. The radius of curvature of a circular cubic is a maximum or a minimum at its intersections with the circles for which it is self-inverse.

Ex. 4. The equation of any circular cubic can be put in the form

$$x(x^2+y^2)+ax^2+2hxy+2gx+2fy=0.$$

[Take the intersection of the cubic with the real asymptote $x=0$ as origin.]

Ex. 5. Through the intersection O of a circular cubic with its real asymptote any chord OPQ of the cubic is drawn. Show that P, Q are equidistant from the singular focus.

[The singular focus of the cubic of Ex. 4 is $(-\frac{1}{2}a,\ -h)$.]

Ex. 6. The normals to a circular cubic with singular focus O at its intersections with a circle for which it is self-inverse touch a parabola with focus O and axis perpendicular to the asymptote.

[The corresponding focal parabola; see Ch. XI, § 11, Ex. 3. Many properties of these normals can be deduced; for instance, the four circles circumscribing the triangles formed by any three of them pass through O and their centres lie on a circle through O, &c.]

Ex. 7. Find the focal parabolas of

$$3x(x^2+y^2)+2x^2+6xy+6y^2=4x$$

and the circles with respect to which it is self inverse.

[The circles have centres $(-1, 1)$, $(-\frac{8}{3}, -4)$, $(1, -\frac{1}{3})$, $(0, 0)$ and radii $\frac{1}{3}\sqrt{30}$, $\frac{1}{3}\sqrt{220}$, $\frac{1}{3}\sqrt{22}$, $\frac{2}{3}\sqrt{(-3)}$.]

Ex. 8. In a one-circuited circular cubic the two real focal parabolas have concavities in opposite directions. In a two-circuited circular cubic the focal parabola through the real foci has its concavity in a direction opposite to that of the other focal parabolas.

Ex. 9. The centres of three circles for which a circular cubic is self-inverse are the vertices of the common self-conjugate triangle of the fourth circle and its focal parabola.

Ex. 10. Two circular cubics can be found with four given concyclic foci, and their real asymptotes are perpendicular.

[Parallel to the axes of the two parabolas through the foci.]

Ex. 11. The lines joining four concyclic foci of a circular cubic in pairs are equally inclined to the real asymptote.

Ex. 12. The centroid of any four concyclic foci of a circular cubic is the foot of the perpendicular from the singular focus on the real asymptote.

[Eliminating y or x from § 3 (ii) and (iii) we get the abscissae or ordinates of the four foci on (iii).]

Ex. 13. The directrices corresponding to four concyclic foci of a circular cubic form a pencil of the same cross-ratio as the pencil subtended by the foci at any point of the focal parabola through them.

[The directrices are found by differentiating (iv) with respect to m, and choosing m so as to make (iv) a line-pair.]

Ex. 14. The centres of the four circles for which a circular cubic is self-inverse lie on a rectangular hyperbola whose asymptotes are the real asymptote of the cubic and the perpendicular from the singular focus.

[The locus of the centre of (iii) is $2(x+p)y+b=0$, if t is taken as a varying parameter.]

Ex. 15. Show that the cubic (i) of § 2 is self-inverse with respect to $x^2+y^2=k$, and that the corresponding focal parabola is

$$(y+h)^2 = b(2x+a).$$

Find an equation giving the other focal parabolas.

Ex. 16. Show that all circular cubics having the same real asymptote and singular focus, and self-inverse with respect to the same point, have a focal parabola in common.

[See Ex. 15.]

Ex. 17. Show that the circle through the real foci of a two-circuited circular cubic meets it in four real points, and that the inverse of the cubic with respect to any one of these points is a circular cubic with an axis of symmetry on which the four real foci lie.

[The corresponding focal parabola meets the circle at the real foci, and the required points are the points of contact with the circle of the common tangents to circle and parabola. The circle inverts into the axis of symmetry.]

Ex. 18. Show that a real circle with respect to which a one-circuited circular cubic is self-inverse meets it in two real points, and that the inverse of the cubic with respect to either of these points is a circular cubic with an axis of symmetry passing through two real foci and bisecting at right angles the line joining the other two real foci.

Ex. 19. A two-circuited circular cubic has $x=0$ as asymptote and three real foci $A(\alpha, 0)$, $B(\beta, 0)$, $C(\gamma, 0)$. Show that its equation is

$$\begin{aligned}(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)\{&8x(x^2+y^2)\\ &+4(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)x-(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)\}\\ &=4\{(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)^2+4\alpha\beta\gamma(\alpha+\beta+\gamma)\}x^2.\end{aligned}$$

[Find the foci of $x(x^2+y^2)+ax+by+c=0$ in the usual manner, and identify them with $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$, $(\delta, 0)$. We have

$$\alpha+\beta+\gamma+\delta=0, \text{ \&c.}]$$

Ex. 20. A circular cubic has three real foci A, B, C on a line meeting the asymptote in H. Prove that

$$(HB^2 - HC^2) \,.\, PA \pm (HC^2 - HA^2) \,.\, PB \pm (HA^2 - HB^2) \,.\, PC = 0,$$

where P is any point on the curve.

$$[(\beta^2-\gamma^2)(x^2+y^2-2\alpha x+\alpha^2)^{\frac{1}{2}} + (\gamma^2-\alpha^2)(x^2+y^2-2\beta x+\beta^2)^{\frac{1}{2}}$$
$$+ (\alpha^2-\beta^2)(x^2+y^2-2\gamma x+\gamma^2)^{\frac{1}{2}} = 0$$

becomes the equation of Ex. 19 when rationalized.]

Ex. 21. Show that, if in Ex. 19 O is the singular focus,

$$OA = -(-\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)^2 \div 4(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta), \text{ \&c.}:$$

and that

$$(OB^{\frac{1}{2}} - OC^{\frac{1}{2}}) \,.\, PA \pm (OC^{\frac{1}{2}} - OA^{\frac{1}{2}}) \,.\, PB \pm (OA^{\frac{1}{2}} - OB^{\frac{1}{2}}) \,.\, PC = 0.$$

Ex. 22. Show that, if in Ex. 20 ABC meets the cubic in U, V, W,

$$8\,HU \,.\, HV \,.\, HW = (HB+HC)(HC+HA)(HA+HB).$$

Ex. 23. A two-circuited circular cubic has real foci A, B, C and the circle ABC meets the cubic in K. The circle of curvature at K meets the circle ABC again at H. Prove that

$$(HB^2 \,.\, KC^2 - HC^2 \,.\, KB^2)\, KA \,.\, PA \pm \ldots \pm \ldots = 0,$$

where P is any point of the curve.

[Invert with respect to K, and we have Ex. 20. Similarly we may invert the results of Ex. 21, 22.]

Ex. 24. A one-circuited circular cubic has $x = 0$ as an asymptote and three real foci $E(\xi, \eta)$, $F(\xi, -\eta)$, $C(\gamma, 0)$. Show that its equation is

$$\xi(\xi^2+\eta^2+\gamma^2+2\xi\gamma)\,\{4x(x^2+y^2)$$
$$+2(3\xi^2-\eta^2+\gamma^2+2\xi\gamma)x - \xi(\xi^2+\eta^2+\gamma^2+2\xi\gamma)\}$$
$$= \{(3\xi^2-\eta^2+\gamma^2+2\xi\gamma)^2 + 4\gamma(\xi^2+\eta^2)(2\xi+\gamma)\}\, x^2.$$

[$(\xi \pm i\eta, 0)$ are foci. Therefore replace α, β by $\xi \pm i\eta$ in Ex. 19.]

Ex. 25. Show that the distances of any point P from the points

$$A(\xi+i\eta, X+iY),\quad B(\xi-i\eta, X-iY),\quad E(\xi+Y, X-\eta),\quad F(\xi-Y, X+\eta)$$

are connected by the relation

$$(\rho+i\sigma)PA + (\rho-i\sigma)PB = \rho(PE^2+PF^2-EF^2+2PE \,.\, PF)^{\frac{1}{2}}$$
$$-\sigma(EF^2-PE^2-PF^2+2PE \,.\, PF)^{\frac{1}{2}},$$

ρ and σ being any real constants.

Ex. 26. Obtain a relation connecting the distances of any point P on the cubic of Ex. 24 from the foci E, F, C by substituting for $PA \pm PB$, &c., from Ex. 25 in Ex. 20 or 21.

By inversion obtain a property of any one-circuited cubic, as in Ex. 23.

Ex. 27. The locus of the foci of a conic through four given concyclic points is a pair of circular cubics with the given points as foci.

[Write down the conditions that the pair of tangents from any point to

$$k(x^2+2fy+a) = y^2+2gx+b$$

should be a circular line-pair, and eliminate k.

The asymptotes of the cubics are parallel to the axes of reference.]

Ex. 28. Any circular cubic passing through its own singular focus O may be considered as the locus of the intersection of any member of a given family of coaxial circles with its diameter through O.

[Take O as origin and the circles as

$$x^2+y^2+px+a = t(py-b).]$$

Ex. 29. Any circular cubic passing through its own singular focus may be considered as the locus of a point P such that $PA \,.\, PB = PC \,.\, PD$, where A, B, C, D are fixed points.

Ex. 30. Find the locus of P if

(i) The tangents of the angles APB, CPD have a constant ratio, A, B, C, D being fixed points.

(ii) The product of the tangents from P to two fixed circles is equal to the product of the tangents from P to two other fixed circles.

(iii) The square of the distance of P from a fixed point multiplied by the distance of P from a fixed line varies as the distance of P from another fixed line.

§ 4. Pencil of Tangents from a Point on a Cubic.

The theorem that the four tangents to a cubic from any point of the curve (excluding the tangent *at* the point) form a pencil of constant cross-ratio was proved in § 3 by means of unreal projection. The theorem is so important that a straightforward proof is given here, only involving real quantities.

It is sufficient to show that, if A and B are any two points on the cubic, the cross-ratio of the pencils of tangents from A and B are the same. Let the tangents at A and B meet at C; and take ABC as the triangle of reference. Since CA, CB are the tangents at A and B, the coefficients of x^3, y^3, x^2z, y^2z in the equation of the cubic are zero.

Hence, choosing suitable real homogeneous coordinates, the equation of the cubic becomes

$$z^3+z^2(x+y)+xy(ax+by+3cz) = 0.$$

The line $x = tz$ meets this cubic where

$$(t+1)z^2+(at^2+2ct+1)yz+bty^2 = 0,$$

and therefore touches it, if

$$(at^2+2ct+1)^2-4bt(t+1) = 0.$$

Hence the tangents from B to the cubic are

$$(ax^2+2cxz+z^2)^2-4bxz^2(x+z) = 0.$$

By Ch. I, § 11, the cross-ratio ϕ of these tangents is given by

$$I^3(\phi+1)^2(\phi-2)^2(\phi-\tfrac{1}{2})^2 = 27J^2(\phi^2-\phi+1)^3,$$

where

$$I = 12\{a^2-ab+b^2+3abc-2(a+b)c^2+c^4\},$$

$$J = 4(2a^2-5ab+2b^2)(a+b)-27a^2b^2+36ab(a+b)c$$
$$-12(2a^2+ab+2b^2)c^2-36abc^3+24(a+b)c^4-8c^6.$$

Since these expressions for I and J are symmetrical in a and b, the cross-ratios of the pencils of tangents from A and B are the same.

Another proof of the theorem is given in Ch. XVI, § 7, Ex. 1.

Ex. 1. Discuss the case tacitly excluded in § 4 in which the tangent at A goes through B, or vice versa.

[Compare the pencils of tangents from A and B with the pencil drawn from any other point of the curve.]

Ex. 2. Show that the pencils of tangents from two adjacent points P and P' of the cubic have the same cross-ratio, by using the fact that the polar conic of P touches the cubic at P and passes through the points of contact of the tangents from P.

Can the theorem of § 4 be deduced?

[Strictly speaking, only if A and B lie on the same circuit of the cubic.]

Ex. 3. If a circular cubic is inverted with respect to a point O on the curve, the cross-ratio of tangents is the same for the curve and its inverse.

By taking O as the intersection of the curve with a real circle for which it is self-inverse, prove that the intersection of two such circles cannot be a focus; and that the four real foci of a harmonic one-circuited cubic lie on a circle for which the cubic is not self-inverse.

[The four tangents from ω invert into tangents from ω', forming a pencil with the same cross-ratio.

The equation of the inverse curve is § 2 (i) with $h=0$ and $k>0$. The real foci are readily found. The curve is harmonic if $ab=2k$, and one-circuited if $4k>a^2$.]

§ 5. The Cubic $(x+y+z)^3+6kxyz=0$.

We proved in § 1 that a non-singular or acnodal cubic has three real collinear inflexions. Choose the tangents at these inflexions as the sides of the triangle of reference, and choose real homogeneous coordinates such that the line of inflexions is $x+y+z=0$.

The most general equation of a cubic is

$$xyz = ax^3+by^3+cz^3+3yz(fy+pz)+3zx(gz+qx)$$
$$+3xy(hx+ry).$$

With our choice of coordinates $x=0$ meets this where $(y+z)^3=0$, so that $b=c=f=p$. Similarly for $y=0$ and $z=0$. Hence:

Any cubic (other than a crunodal cubic, a cuspidal cubic, or a cubic with three concurrent real inflexional tangents) can be put in the form

$$(x+y+z)^3+6kxyz=0$$

by a real choice of homogeneous coordinates.

Conversely, the cubic

$$f \equiv (x+y+z)^3+6kxyz=0 \quad . \quad . \quad . \quad . \quad . \quad (\text{i})$$

has three and only three real inflexions $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$, the real inflexional tangents $x=0$, $y=0$, $z=0$ being not concurrent. Hence we cannot put the equation of a cubic with three real concurrent inflexional tangents or of a cubic with a cusp or crunode into the form (i).

The curve (i) has a double point when

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0.$$

This gives, on excluding $x=y=z=0$,

$$k=-\tfrac{9}{2} \text{ and } x=y=z.$$

Hence:

An acnodal cubic can be put in the form

$$(x+y+z)^3=27xyz$$

by a real choice of homogeneous coordinates, the acnode being the point $(1, 1, 1)$.

To find the cross-ratio ϕ of the pencil of tangents drawn to the cubic from any point of the curve, it is sufficient to consider the tangents drawn from the point $(1, -1, 0)$.

Now $t(x+y)=z$ meets the curve where

$$(t+1)^3(x+y)^2+6ktxy=0,$$

and therefore touches it if

$$4t^4+12t^3+6(2+k)t^2+4t=0.$$

Proceeding now as in § 4 we have

$$k(k+4)^3\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2=(k^2+6k+6)^2(\phi^2-\phi+1)^3.$$

The tangential equation of the cubic, found as in Ch. IV, § 3, is

$$9(\mu-\nu)^2(\nu-\lambda)^2(\lambda-\mu)^2+4k(-2\mu\nu+\nu\lambda+\lambda\mu)$$
$$(\mu\nu-2\nu\lambda+\lambda\mu)(\mu\nu+\nu\lambda-2\lambda\mu)-12k^2\lambda^2\mu^2\nu^2=0.$$

Ex. 1. For what value of k is the pencil of tangents from any point of the cubic $(x+y+z)^3+6kxyz=0$ (i) harmonic, (ii) equianharmonic?

[(i) $k^2+6k+6=0$, (ii) $k=-4$.]

Ex. 2. Find the inflexions of the cubic

$$f \equiv (x+y+z)^3+6kxyz=0.$$

[The Hessian is

$$H \equiv (x+y+z)(2yz+2zx+2xy-x^2-y^2-z^2)+2kxyz=0.$$

The inflexions lie on the lines

$$f-3H \equiv 4(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)=0.]$$

Ex. 3. Show that in general two cubics with given inflexional tangents at three given collinear inflexions touch a given line. Mention any exceptions.

[If the line passes through an inflexion or the intersection of two of the given tangents, there is only one cubic.]

Ex. 4. Given the tangents at three given collinear inflexions of a cubic, the locus of the remaining inflexions is two straight lines.

[See Ex. 2.]

Ex. 5. Show that nine cubics can be drawn in general to pass through three given points, given three inflexional tangents and the fact that the three corresponding inflexions are collinear.

[If $l_i x+m_i y+n_i z=0$ $(i=1, 2, 3)$ are the inflexional tangents and the three given points are the vertices of the triangle of reference, the cubics are

$$(l_1 x+m_1 y+n_1 z)(l_2 x+m_2 y+n_2 z)(l_3 x+m_3 y+n_3 z) = \{(l_1 l_2 l_3)^{\frac{1}{3}} x+(m_1 m_2 m_3)^{\frac{1}{3}} y+(n_1 n_2 n_3)^{\frac{1}{3}} z\}^3,$$

any value of the cube roots being taken.]

Ex. 6. Under what circumstances is the triangle formed by the real inflexional tangents of a cubic self-conjugate with respect to one of the polar conics of the cubics?

[The polar conic of (1, 1, 1), when $k=-3$.]

Ex. 7. The point and tangential equations of an acnodal cubic can be put in the form

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0, \qquad \lambda^{-\frac{1}{2}}+\mu^{-\frac{1}{2}}+\nu^{-\frac{1}{2}}=0.$$

§ 6. Symmetry of Cubics.

A curve of the second degree has always symmetry, but this is not true of a curve of the third degree. However, a cubic may always be projected into a symmetrical curve.

Firstly, we have:

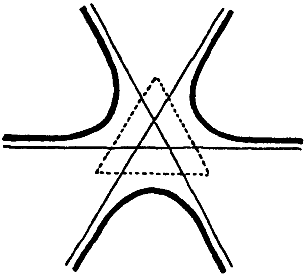
A cubic can be projected so as to be its own reflexion in a line.

For suppose I is any real inflexion and l its harmonic polar.

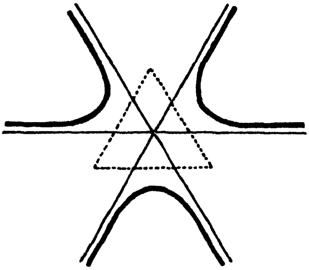
Let any line through I meet the cubic again in P, Q and l in R. Then (IR, PQ) is harmonic.

Hence, if we project I to infinity and the angle between l and IR into 90°, PQ will be bisected at right angles at R. Again,

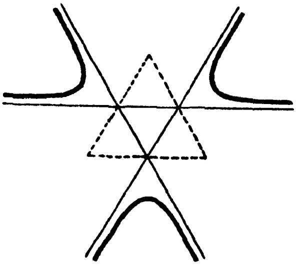
A cubic can be projected so as to be symmetrical about a centre.



$m = \frac{3}{2}$, $h = \frac{1}{8}$.
Fig. 4.



$m = 1$.
Fig. 5.



$m = \frac{1}{2}$, $h = -\frac{1}{2}$.
Fig. 6.

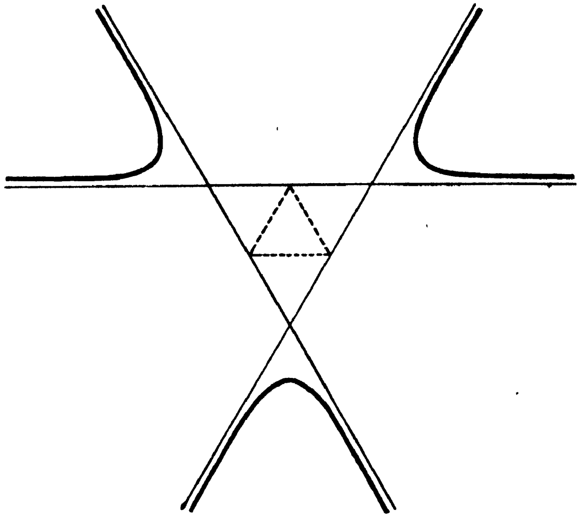
For if we project l to infinity, I will be the middle point of PQ.

These two theorems hold for nodal and cuspidal cubics also.

More interesting, perhaps, is the result due to Clifford (*Collected Works*, p. 412):

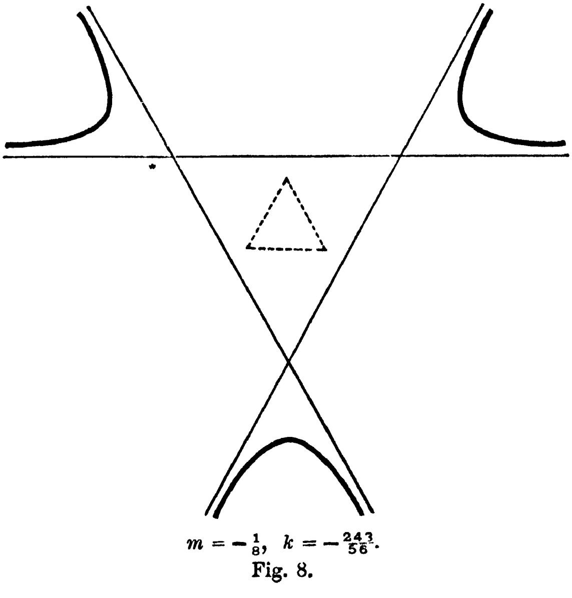
Any cubic (other than a crunodal or cuspidal cubic) can be projected so as to have the symmetry of an equilateral triangle.

In fact project the cubic so that the line of real inflexions is at infinity, and the triangle formed by the real inflexional



$m = 0, \quad k = -4.$

Fig. 7.



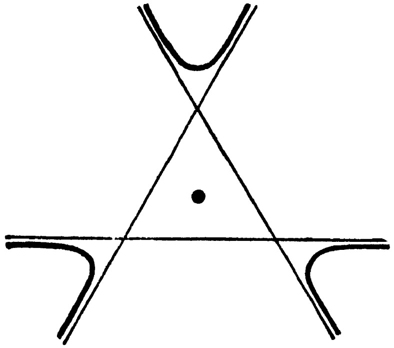
$m = -\frac{1}{8}, \quad k = -\frac{243}{56}.$

Fig. 8.

tangents is equilateral. Then, taking areal coordinates, we proved in § 5 that the equation becomes

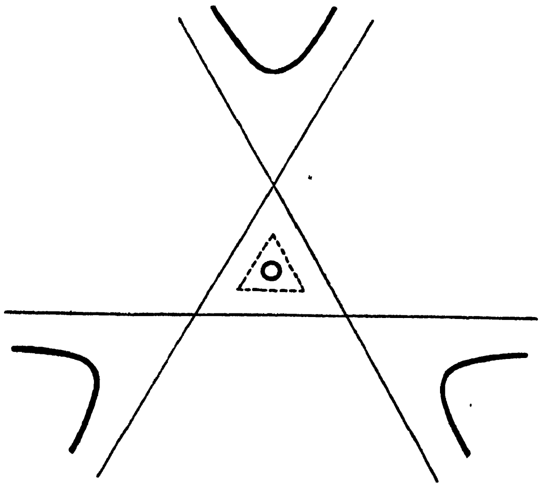
$$(x+y+z)^3+6kxyz = 0,$$

from which the symmetry is obvious.



$k = -\frac{9}{2}$.

Fig. 9.



$m = -1, \quad k = -\frac{32}{7}$.

Fig. 10.

The case in which the inflexional tangents are concurrent wants separate discussion.

In this case project the curve so that the line of real inflexions is at infinity and the angles between the inflexional tangents are 120°. Take their intersection as origin and one of them as the axis of x. Then the asymptotes of the curve are $y=0$, $y=\pm\sqrt{3}\,.\,x$; and since the asymptotes meet the curve in no finite point the equation of the curve is of the form

$$y(3x^2-y^2)=a^3,$$

or in polar coordinates

$$r^3\sin 3\theta = a^3,$$

from which the symmetry is obvious.

The cubic has one or two circuits according as $y=z$ meets the curve in one or three real points, i. e. according as k is greater than or less than $-\frac{9}{2}$.

Diagrams of the cubic for various values of k are given in Figures 4 to 10. The m and the dotted lines are explained in § 7. We have $(4m^2-2m+1)k=4(m-1)^3$.

§ 7. The Cubic $x^3+y^3+z^3+6mxyz=0$.

Every real non-singular cubic can be put in the form

$$x^3+y^3+z^3+6mxyz=0$$

by a real choice of homogeneous coordinates.

We proved in § 5 that a cubic can be put in the form

$$(x+y+z)^3+6kxyz=0 \quad \ldots\ldots \quad \text{(i)}$$

by a real choice of homogeneous coordinates.

In this equation put

$$x=-2mX+Y+Z,\ y=X-2mY+Z,\ z=X+Y-2mZ,$$

where $$4(m-1)^3=k(4m^2-2m+1) \quad \ldots\ldots \quad \text{(ii)}$$

i. e. $$2(1-m)(2m+1)X=(2m+1)x+y+z, \text{ \&c.}$$

This will be lawful except in the cases

$$m=1 \text{ when } k=0, \text{ and } m=-\tfrac{1}{2} \text{ when } k=-\tfrac{9}{2}.$$

Then straightforward reduction gives

$$(x+y+z)^3+6kxyz \equiv \frac{(2m+1)^2}{4m^2-2m+1}(X^3+Y^3+Z^3+6mXYZ).$$

Hence if we choose m as the real root of (ii), i. e.

$$m=\{2(2k+9)^{\frac{1}{3}}+(2k)^{\frac{1}{3}}\} \div \{2(2k+9)^{\frac{1}{3}}-2(2k)^{\frac{1}{3}}\},$$

and replace X, Y, Z by x, y, z * we have the cubic in the required form.

We now consider the cases $m=1$ and $m=-\frac{1}{2}$.

Firstly, we note that, if

$$f \equiv x^3+y^3+z^3+6mxyz=0 \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

has a double point, $\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=\dfrac{\partial f}{\partial z}=0$ at the double point, which gives on elimination of $x:y:z$ that $8m^3+1=0$. But a value of m satisfying this equation makes f split up into three linear factors.† Hence (iii) cannot be a unicursal cubic unless it is three straight lines. This explains the failure of the above argument when $k=-\frac{9}{2}$, which gives us an acnodal cubic (§ 5).

Secondly, we note that our argument does not apply to the cubic for which the tangents at the three real collinear inflexions are concurrent; for such a cubic cannot be put in the form (i). We showed in § 6 that it can be put in the form

$$y(3x^2-y^2)=a^3;$$

and putting

$$X=-2^{\frac{1}{3}}a+3^{\frac{1}{2}}x+y,\ Y=-2^{\frac{1}{3}}a-3^{\frac{1}{2}}x+y,\ Z=-2^{\frac{1}{3}}a-2y,$$

we have

$$X^3+Y^3+Z^3+6XYZ=-18a^3-18y^3+54x^2y=0.$$

Hence the case $m=1$ gives us the cubic in which the tangents at three real collinear inflexions are concurrent.

The sides of the triangle of reference for the cubic (iii) are given by the dotted lines of Figures 4 to 10.

§ 8.

The inflexions of

$$x^3+y^3+z^3+6mxyz=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

are its intersections with the sides of the triangle of reference. In fact the equation may be written

$$(-2mx+y+z)(-2mx+\omega^2y+\omega z)(-2mx+\omega y+\omega^2z) \\ +(8m^3+1)x^3=0;$$

which shows that

$$-2mx+y+z=0,\quad -2mx+\omega^2y+\omega z=0, \\ -2mx+\omega y+\omega^2z=0$$

are inflexional tangents whose points of contact lie on $x=0$.

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* By Ch. I, § 1, this is equivalent to choosing fresh homogeneous coordinates with $(2m+1)x+y+z=0$, &c., as the sides of the new triangle of reference.

† See the beginning of § 8.

Another proof is obtained by writing down the polar conic of any intersection of the cubic and $x=0$, for instance $(0, -1, 1)$. It will be found to be

$$(-2mx+y+z)(y-z)=0.$$

Since this degenerates, $(0, -1, 1)$ is an inflexion,

$$-2mx+y+z=0$$

being the inflexional tangent and $y=z$ the harmonic polar of the inflexion. Another proof is given in § 9.

The nine inflexions of (i) are given by the scheme

$$\left.\begin{array}{lll} (0, -1, 1), & (0, -\omega^2, \omega), & (0, -\omega, \omega^2) \\ (1, 0, -1), & (\omega, 0, -\omega^2), & (\omega^2, 0, -\omega) \\ (-1, 1, 0), & (-\omega^2, \omega, 0), & (-\omega, \omega^2, 0) \end{array}\right\}.$$

The inflexional tangents are

$$\left.\begin{array}{lll} 2mx+y+z=0, & -2mx+\omega y+\omega^2 z=0, & -2mx+\omega^2 y+\omega z=0 \\ x-2my+z=0, & \omega^2 x-2my+\omega z=0, & \omega x-2my+\omega^2 z=0 \\ x+y-2mz=0, & \omega x+\omega^2 y-2mz=0, & \omega^2 x+\omega y-2mz=0 \end{array}\right\}$$

The harmonic polars of the inflexions are

$$\left.\begin{array}{lll} y=z, & y=\omega z, & y=\omega^2 z \\ z=x, & z=\omega x, & z=\omega^2 x \\ x=y, & x=\omega y, & x=\omega^2 y \end{array}\right\}.$$

The inflexions lie by threes on twelve lines

$$\left.\begin{array}{r} x=0 \\ y=0 \\ z=0 \end{array}\right\}, \quad \left.\begin{array}{r} x+y+z=0 \\ x+\omega y+\omega^2 z=0 \\ x+\omega^2 y+\omega z=0 \end{array}\right\} \quad \left.\begin{array}{r} \omega x+y+z=0 \\ x+\omega y+z=0 \\ x+y+\omega z=0 \end{array}\right\}, \quad \left.\begin{array}{r} \omega^2 x+y+z=0 \\ x+\omega^2 y+z=0 \\ x+y+\omega^2 z=0 \end{array}\right\}.$$

Three inflexions are collinear if (1) they lie in the same row of the inflexions-scheme, (2) they lie in the same column, (3) no two of the three inflexions lie either in the same row or in the same column.

The twelve lines intersect in twenty-one points, namely the nine inflexions and the twelve '*critic centres*' of the cubic

$$\left.\begin{array}{llll} (1, 0, 0), & (1, 1, 1), & (\omega^2, 1, 1), & (\omega, 1, 1) \\ (0, 1, 0), & (1, \omega^2, \omega), & (1, \omega^2, 1), & (1, \omega, 1) \\ (0, 0, 1), & (1, \omega, \omega^2), & (1, 1, \omega^2), & (1, 1, \omega) \end{array}\right\}.$$

If ϕ is the cross-ratio of the pencil of tangents from any point of (i) to the curve,

$$64m^3(m^3-1)^3\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2 = (8m^6+20m^3-1)^2(\phi^2-\phi+1)^3.$$

This may be proved as in § 5 by considering the pencil of tangents from $(-1, 1, 0)$ to the curve; or by putting

$$k = 4(m-1)^3 \div (4m^2-2m+1) \quad . \quad . \quad . \quad \text{(ii)}$$

in the expression for the cross-ratio found in that section.*

The tangential equation of the curve is

$$\lambda^6+\mu^6+\nu^6-(32m^3+2)(\mu^3\nu^3+\nu^3\lambda^3+\lambda^3\mu^3)$$
$$-24m^2\lambda\mu\nu(\lambda^3+\mu^3+\nu^3)-(48m^4+24m)\lambda^2\mu^2\nu^2=0.$$

The curve has two circuits if $m < -\frac{1}{2}$, and one circuit if $m > -\frac{1}{2}$, as may be deduced from § 6 by means of equation (ii) above.

As an exercise the reader may obtain this result from the expression just obtained for ϕ, noting that ϕ is only real (and not -1, $\frac{1}{2}$, or 2), if the cubic has two circuits.

Ex. 1. Show that the equation of any cubic, not crunodal or cuspidal, can be put in the form

$$y^2z+yz^2+z^2x+zx^2+x^2y+xy^2+6lxyz=0.$$

Find the cross-ratio of the tangents from any point of the curve.

[Consider the tangents from $(1, 0, 0)$. The cubic is acnodal, if $l=-1$.]

Ex. 2. Find the cross-ratio of the tangents from any point of

$$y^2z+x^2y+z^2x+6pxyz=0.$$

[Consider the tangents from $(0, 0, 1)$. The pencil of tangents is harmonic if $216p^6+36p^3+1=0$, and equianharmonic if $p=0$ or $9p^3+1=0$. We show in Ch. XVI, § 6, Ex. 15, that the equation of a cubic can be put into this form in twenty-four different ways.]

Ex. 3. Show that the equation of any cubic, not crunodal or cuspidal or having three real concurrent inflexional tangents, can be put in the form

$$x^3+y^3+z^3=h(x+y+z)^3.$$

[Replace x, y, z by $y+z$, $z+x$, $x+y$ in § 7 (i).]

Ex. 4. The inflexions of the cubic of Ex. 3 lie on

$$(x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z)=0.$$

The lines $x=0$, $y=0$, $z=0$ each pass through a real inflexion, the inflexional tangents being $y+z=0$, &c.

Ex. 5. Find the cross-ratio of the pencil formed by the tangents from any point of the cubic of Ex. 3.

[Consider the tangents from $(-1, 1, 0)$.]

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* For the case of any cubic, see Elliott's *Algebra of Quantics*, § 291, &c.

§ 9. **Syzygetic Cubics.**

The Hessian of

$$x^3+y^3+z^3+6mxyz = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

is at once found to be

$$x^3+y^3+z^3+6Mxyz = 0$$

where

$$M = -(2m^3+1)/6m^2 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

The curve and its Hessian meet where $xyz = 0$; i. e. the inflexions of the cubic lie on the sides of the triangle of reference, as proved before.

We see that:

A cubic has the same inflexions as its Hessian.

The family of cubics found by varying m in the equation (i) all have the same inflexions; they are called a family of *syzygetic cubics.* The Hessian of any member of the family belongs to the family.

The harmonic polars of the inflexions are the same for all the syzygetic cubics of the family.

Since (ii) is of the third degree when considered as an equation in m,

Any non-singular cubic is the Hessian of three syzygetic cubics.

Ex. 1. A real cubic is the Hessian of one or three real cubics according as it has one or two circuits.

[The roots of the equation $2m^3+6Mm^2+1 = 0$ in m are all real, if $8M^3+1 < 0$.]

Ex. 2. All cubics through eight given inflexions of a cubic pass through the ninth inflexion and are syzygetic with it.

[If $f = 0$ is the cubic and $H = 0$ its Hessian, any cubic through eight inflexions is $f+kH = 0$].

Ex. 3. The polar line of a point P on a cubic with respect to any syzygetic cubic passes through the tangential of P.

[If P is (x, y, z), the tangential of P is

$$(xy^3-xz^3,\ yz^3-yx^3,\ zx^3-zy^3).]$$

Ex. 4. The polar conic of a point P with respect to any cubic of a syzygetic family goes through four fixed points lying on that cubic of the family which passes through P.

[If P is (ξ, η, ζ), the polar conic passes through the four intersections of

$$\xi x^2+\eta y^2+\zeta z^2 = 0 \qquad \text{and} \qquad \xi yz+\eta zx+\zeta xy = 0.]$$

Ex. 5. If the pencil of tangents from any point of a cubic is harmonic, the cubic is the Hessian of its Hessian.

[$8m^6+20m^3-1 = 0$. We may call the cubic 'harmonic'.]

Ex. 6. If the pencil of tangents from any point of a cubic is equianharmonic, the Hessian degenerates.

[$m^3(m^3-1)=0$. We may call the cubic 'equianharmonic'.]

Ex. 7. There are two species of equianharmonic cubic. One species has a Hessian consisting of three real lines. Each real inflexional tangent meets two unreal inflexional tangents at a critic centre which is a vertex of the triangle formed by the Hessian.

The other species has a Hessian consisting of one real and two unreal lines. Its real inflexional tangents are concurrent at a critic centre.

[$m=0$ and $m^3=1$ in the two cases. The critic centres are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$ respectively.]

Ex. 8. Under what conditions is the polar conic of a point P for a cubic a coincident line-pair?

[Either (1) the cubic is cuspidal, and P is the cusp or the intersection of the cuspidal and inflexional tangents, or (2) the cubic is equianharmonic, and P is one of the critic centres referred to in Ex. 7.]

Ex. 9. If all polar conics for a cubic have a common self-conjugate triangle, the cubic is equianharmonic. The sides of the triangle are the degenerate Hessian.

Obtain the equations of the sides of the triangle for the cubic of § 8, Ex. 2, in the cases $p=0$ and $p=-3^{-\frac{2}{3}}$; show that the cubic is equianharmonic of the second and first species respectively.

Ex. 10. The tangents at two inflexions of a cubic meet on the harmonic polar of the inflexion collinear with the two given inflexions.

Ex. 11. The intersection of the tangent at an inflexion of a cubic with its harmonic polar lies on the Hessian.

[If $(0, -1, 1)$ is the inflexion, the intersection is $(1, m, m)$.]

Ex. 12. The polar l of P with respect to the polar conic of Q for a cubic is the same as the polar of Q with respect to the polar conic of P.

If P and Q are fixed and the cubic is any member of a syzygetic family, l goes through a fixed point. If this point is on PQ, P and Q lie on the same cubic of the syzygetic family.

If R is any point on l, the relation between P, Q, R is symmetric.

[PQR is an 'apolar' triangle of the 3-ic. If it is taken as triangle of reference, the coefficient of xyz in the equation of the 3-ic is zero.]

Ex. 13. If any line meets

$$x^3+y^3+z^3+6mxyz=0$$

in (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),

then $$x_1x_2x_3+y_1y_2y_3+z_1z_2z_3=0,$$

$$x_1y_2z_3+x_1y_3z_2+x_2y_3z_1+x_2y_1z_3+x_3y_1z_2+x_3y_2z_1=0.$$

[See Ch. I, § 6, Ex. 8, and Ch. VII, § 3, Ex. 15. Note list of 'Errata'.]

Ex. 14. The inflexions of a cubic lie by threes on the lines a, b, c. The tangents at the inflexions on b and the tangents at the inflexions on c form two triangles whose vertices lie on a conic and whose sides touch a conic. Both conics touch b and c at their intersections with a.

[If a, b, c are $x=0$, $y=0$, $z=0$, the conics are

$$x^2+2myz=0 \quad \text{and} \quad x^2+8myz=0.]$$

Ex. 15. Two triangles are such that each side of either triangle passes through three inflexions of a cubic, and no two sides of the same triangle pass through the same inflexion. Show that the two triangles are in perspective in six ways, and that the axes of perspective each pass through three inflexions.

[If the sides of the triangles are $x = 0$, $y = 0$, $z = 0$, and

$$x+y+z=0, \quad x+\omega y+\omega^2 z=0, \quad x+\omega^2 y+\omega z=0,$$

the axes of perspective are $\omega x+y+z=0$, &c.]

Ex. 16. The pencil of four lines joining an inflexion of a cubic with the other eight inflexions is equianharmonic.

[If the inflexion is $(0, -1, 1)$, the lines meet $z=0$ at

$$(0, 1, 0), \quad (-1, 1, 0), \quad (-\omega^2, 1, 0), \quad (-\omega, 1, 0).]$$

Ex. 17. Two lines each passing through three inflexions of a cubic meet at O. Show that the points of contact of the tangents from O lie by pairs on three lines through the remaining inflexions.

[$x^3+y^3+z^3+6mxyz+kx(x^2+2myz)=0$ is a line-trio, if

$$8m^3(k+3)^3+27(k+1)=0.]$$

Ex. 18. The three intersections of the line through three collinear inflexions with their harmonic polars form an involution with those inflexions.

[If the inflexions are $(0, -1, 1)$, $(1, 0, -1)$, $(-1, 1, 0)$, the intersections of $x+y+z=0$ with their harmonic polars are

$$(-2, 1, 1), \quad (1, -2, 1), \quad (1, 1, -2).$$

The double points of the involution are $(\omega, \omega^2, 1)$, $(\omega^2, \omega, 1)$.]

Ex. 19. The harmonic polars of three collinear inflexions of a cubic pass through the same critic centre A, and the remaining inflexions lie on two lines through A.

The line joining A to any one of the collinear inflexions is divided harmonically by the cubic.

[Take the collinear inflexions as those on $x=0$. Then A is $(1, 0, 0)$.]

Ex. 20. The three real lines each passing through one real and two unreal inflexions of a two-circuited cubic form a triangle. Show that from each vertex of the triangle two real tangents can be drawn, that the six points of contact lie on a conic, that the tangents meet the cubic again in six points on a conic, and that the six tangents touch a conic.

[Ex. 20 to 24 are obvious from Fig. 4 to 10.]

Ex. 21. From each intersection of two real inflexional tangents of a two-circuited cubic two real tangents can be drawn to the oval. Show that these six tangents touch a conic, that their points of contact lie on a conic, and that the intersections of each pair of tangents with the third inflexional tangent lie on a conic.

Ex. 22. The three lines joining two real critic centres and passing through a real inflexion meet the real inflexional tangents at those inflexions and at six points lying on a conic.

[All the conics of Ex. 20, 21, 22 pass through the same two unreal points on the line of real inflexions.]

Ex. 23. The six tangents drawn to a cubic from a point on the harmonic polar of an inflexion I form an involution, and the chords of contact of conjugate pairs of the involution pass through I.

[There are also twelve cubics such that the tangents from any point on one of them to the given cubic form an involution. Each such cubic has three collinear inflexions and corresponding inflexional tangents in common with the given cubic. See Roberts, *Proc. London Math. Soc.*, xiii (1882), p. 26.]

Ex. 24. It is possible to project a family of syzygetic cubics so that each curve has the symmetry of an equilateral triangle.

Ex 25. The 'polo-conic' of the lines l and l' with respect to a cubic is the locus of a point whose polar conic has l and l' as conjugate lines. The 'polo-conic' of l is the locus of a point whose polar conic touches $l\,(l' \equiv l)$. Show that

(i) The polo-conic of l and l' is the locus of the pole of l with respect to the polar conic of any point on l', and vice versa.

(ii) The polar line of any point on l touches the polo-conic of l.

(iii) The polo-conic of l touches the Hessian at three points, which lie on the polo-conic of l and l'.

[The polo-conic of

$$\lambda x+\mu y+\nu z=0 \qquad \text{and} \qquad \lambda' x+\mu' y+\nu' z=0$$

is $\lambda\lambda'(m^2x^2-yz)+\ldots+\ldots+(\mu\nu'+\mu'\nu)(mx^2-m^2yz)+\ldots+\ldots=0.$]

Ex. 26. The configuration formed by the nine inflexions and the twelve lines joining them is dualistic to the configuration formed by the nine harmonic polars of the inflexions and the twelve critic centres.]

Ex. 27. If the equation $f=0$ of a cubic is given, prove that the inflexions can be found without solving any equation of degree higher than 4.

[Let $H=0$ be the Hessian of $f=0$. Find k such that $f+kH\equiv 0$ coincides with its Hessian; then it will be a line-trio on which the inflexions lie.]

Ex. 28. A quartic through the twelve critic centres of a cubic has an inflexion at each critic centre.

[The quartic is $ax(y^3-z^3)+by(z^3-x^3)+cz(x^3-y^3)=0.$]

Ex. 29. A doubly infinite family of quintics passes through the nine inflexions and the twelve critic centres of a cubic. Each such quintic has an inflexion at every inflexion of the cubic, and the nine inflexional tangents are concurrent at a point of the quintic.

[The quintic is

$$ayz(y^3+z^3-2x^3)+bzx(z^3+x^3-2y^3)+cxy(x^3+y^3-2z^3)=0.$$

The inflexional tangents are

$$(b+c)x=a(y+z), \qquad (b+c\omega)x=a(y+\omega z), \qquad (b+c\omega^2)x=a(y+\omega^2 z),$$

&c., meeting at (a, b, c).]

Ex. 30. Defining a syzygetic family of cubics by the equation

$$f+kH=0,$$

where $f=0$ is a cubic and $H=0$ its Hessian, show that

(i) A syzygetic family of nodal cubics may be projected so as to be all similar and similarly situated with the node as centre of similitude, and

(ii) A syzygetic family of cuspidal cubics may be projected so as to be all congruent and have a common asymptote which is the cuspidal tangent of each curve.

[(i) See Ch. XIII, § 4, Ex. 15. (ii) Project into $(x-k)y^2=1$.]

CHAPTER XV

CUBICS AS JACOBIANS

§ 1. Jacobian of three Conics.

SUPPOSE that $U = 0$, $V = 0$, $W = 0$ are the equations in homogeneous coordinates of three conics not passing through the same four points. The polars of the point $P(x', y', z')$ with respect to the conics are

$$x\frac{\partial U}{\partial x'} + y\frac{\partial U}{\partial y'} + z\frac{\partial U}{\partial z'} = 0$$

and two similar equations; where $\frac{\partial U}{\partial x'}$ means the result of putting x' for x, y' for y, z' for z in $\frac{\partial U}{\partial x}$, &c.

These polars are concurrent (at Q say) if

$$\begin{vmatrix} \frac{\partial U}{\partial x'} & \frac{\partial U}{\partial y'} & \frac{\partial U}{\partial z'} \\ \frac{\partial V}{\partial x'} & \frac{\partial V}{\partial y'} & \frac{\partial V}{\partial z'} \\ \frac{\partial W}{\partial x'} & \frac{\partial W}{\partial y'} & \frac{\partial W}{\partial z'} \end{vmatrix} = 0.$$

Dropping the dashes we have the equation of the locus of a point whose polars with respect to the three given conics are concurrent. It is evidently a cubic curve, called the *Jacobian* of the conics.* Its equation is usually written

$$\frac{\partial(U.\ V,\ W)}{\partial(x,\ y,\ z)} = 0.$$

Since the polar of P with respect to each conic passes through Q, the polar of Q passes through P. Hence Q is also on the Jacobian. We call P and Q conjugate points on the Jacobian.

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* For the more general case see Ch. VII, § 10.

The Jacobian of the conics

$$\lambda_1 U+\mu_1 V+\nu_1 W=0,\quad \lambda_2 U+\mu_2 V+\nu_2 W=0,$$
$$\lambda_3 U+\mu_3 V+\nu_3 W=0,$$

where the λ's, μ's, ν's, are constants, is

$$0=\frac{\partial(\lambda_1 U+\mu_1 V+\nu_1 W,\ \lambda_2 U+\mu_2 V+\nu_2 W,\ \lambda_3 U+\mu_3 V+\nu_3 W)}{\partial(x,y,z)}$$

$$\equiv\begin{vmatrix}\lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3\end{vmatrix}\times\frac{\partial(U,V,W)}{\partial(x,y,z)};$$

as is at once verified by the rule for multiplying determinants.

If these conics all pass through the same four points,

$$\begin{vmatrix}\lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3\end{vmatrix}=0.$$

In this case the equation of their Jacobian vanishes identically; as is geometrically obvious, for the polars of *any* point with respect to all conics through four given points are concurrent.

If the conics do not pass through the same four points, their Jacobian is identical with the Jacobian of the original conics $U=0$, $V=0$, $W=0$. Hence:

The Jacobian of any three conics of the family

$$\lambda U+\mu V+\nu W=0$$

(which do not pass through the same four points) is identical with the Jacobian of $U=0$, $V=0$, $W=0$.

The conic $\lambda U+\mu V+\nu W=0$ has a node at (X, Y, Z), i.e. is a line-pair meeting at (X, Y, Z), if

$$\lambda\frac{\partial U}{\partial X}+\mu\frac{\partial V}{\partial X}+\nu\frac{\partial W}{\partial X}=0,\quad \lambda\frac{\partial U}{\partial Y}+\mu\frac{\partial V}{\partial Y}+\nu\frac{\partial W}{\partial Y}=0,$$
$$\lambda\frac{\partial U}{\partial Z}+\mu\frac{\partial V}{\partial Z}+\nu\frac{\partial W}{\partial Z}=0.$$

Eliminating $\lambda:\mu:\nu$ between these equations, we see that (X, Y, Z) must lie on the Jacobian; and if this is the case, the equations give one and only one value of the ratio $\lambda:\mu:\nu$.

Hence:

Any point on the Jacobian of a family of conics is the centre of one and only one degenerate conic of the family, and the centre of any degenerate conic must lie on the Jacobian.

If P, Q are conjugate points on the Jacobian, and A is any other point on the Jacobian, the lines AP and AQ are harmonic conjugates with respect to the degenerate conic with centre A. For AQ is the polar of P with respect to the degenerate conic.

Ex. 1. If three conics have a point A in common, their Jacobian has a node at A.

[Take A as $(1, 0, 0)$.]

Ex. 2. If three conics have points B, C in common, their Jacobian degenerates into the line BC and a conic through B and C.

Ex. 3. The Jacobian of three circles is the line at infinity and the circle orthogonal to them all.

[Take the circles as

$$x^2+y^2+2g_i x+2f_i y+c=0, \quad (i=1, 2, 3).]$$

Ex. 4. Find the circle orthogonal to

$$x^2+y^2+8x+15=0, \qquad x^2+y^2+2x-12y+21=0,$$
$$x^2+y^2-2x+4y+1=0.$$

$[x^2+y^2+2x-2y-7=0.]$

Ex. 5. If three conics pass through the points A, B, C, their Jacobian degenerates into the lines BC, CA, AB.

Ex. 6. If three conics pass through A and two touch at A, their Jacobian has a node at A and one branch of the Jacobian touches the two conics at A.

Ex. 7. If three conics have a common pole and polar, their Jacobian degenerates.

[Into the common polar and two lines through the pole.]

Ex. 8. If three conics have a common self-conjugate triangle, their Jacobian degenerates into the sides of that triangle.

[See Ex. 7.]

Ex. 9. The sides of the common self-conjugate triangle of any two conics may be obtained by forming the Jacobian of the two conics and their harmonic locus.

[The two conics and their harmonic locus have a common self-conjugate triangle. See Ch. IV, § 4, Ex. 4.]

Ex. 10. The Jacobian of two conics and a coincident line-pair degenerates.

[Into the line and a conic.]

Ex. 11. The Jacobian of a conic, a circle with centre P, and a coincident line-pair at infinity is the rectangular hyperbola through the feet of the normals from P to the conic together with the line at infinity.]

[The Jacobian of

$$ax^2+by^2=z^2, \qquad x^2+y^2-2\xi xy-2\eta yz+cz^2=0, \qquad z^2=0$$

is

$$by(x-\xi z)=ax(y-\eta z).$$

Now put $z=1$.]

Ex. 12. The axes of a conic (using any coordinates) may be obtained by writing down the Jacobian of the conic, its director circle, and a coincident line-pair at infinity.

[We suppose that we know the equation of *one* circle and of the line at infinity. We thus obtain the tangential equation of the circular points. Then, forming the harmonic locus of the conic and the circular points, we have the director circle. The Jacobian degenerates into the axes and the line at infinity by Ex. 11.]

Ex. 13. The Jacobian of two conics having double contact and any third conic degenerates.

[Into the chord of contact and a conic dividing the chord of contact harmonically.]

Ex. 14. The Jacobian of two concentric circles and any conic is a rectangular hyperbola and the line at infinity.

[See Ex. 13. Discuss the case in which the conic is a circle.]

Ex. 15. If the Jacobian of three conics passes through the intersection A of two of them, the two conics touch at A.

[Take A as the point $(1, 0, 0)$.]

Ex. 16. The locus of the point of contact of two members of the family $\lambda U+\mu V+\nu W=0$ is their Jacobian.

[See Ex. 15.]

§ 2. Any Cubic as a Jacobian of three Conics.

Suppose $f(x, y, z) = 0$ is the equation of a cubic. Then the polar conic, i.e. the first polar curve, of the point $P\,(\xi, \eta, \zeta)$ is

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

The Jacobian of any three polar conics is the same as the Jacobian of

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0, \quad \frac{\partial f}{\partial z}=0,$$

and is therefore

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial x\,\partial z} \\ \dfrac{\partial^2 f}{\partial y\,\partial x} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial y\,\partial z} \\ \dfrac{\partial^2 f}{\partial z\,\partial x} & \dfrac{\partial^2 f}{\partial z\,\partial y} & \dfrac{\partial^2 f}{\partial z^2} \end{vmatrix} = 0,$$

which is the Hessian of $f(x, y, z) = 0$. Now any cubic* is the Hessian of three different cubics by Ch. XIV, § 9. Hence:

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* i e. non-singular cubic. See § 3, Ex. 9.

Any cubic may be considered as the Jacobian of a family of conics, and that in three different ways.

Suppose now that the polars of P with respect to

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

are concurrent at the point $Q\,(X, Y, Z)$, so that P and Q are conjugate points of the Jacobian. Then

$$\begin{aligned}
\xi \frac{\partial^2 f}{\partial X^2} + \eta \frac{\partial^2 f}{\partial X \partial Y} + \zeta \frac{\partial^2 f}{\partial X \partial Z} &= 0, \\
\xi \frac{\partial^2 f}{\partial Y \partial X} + \eta \frac{\partial^2 f}{\partial Y^2} + \zeta \frac{\partial^2 f}{\partial Y \partial Z} &= 0, \\
\xi \frac{\partial^2 f}{\partial Z \partial X} + \eta \frac{\partial^2 f}{\partial Z \partial Y} + \zeta \frac{\partial^2 f}{\partial Z^2} &= 0 \quad . \; . \; . \quad \text{(ii)}.
\end{aligned}$$

But equations (ii) are the conditions that Q should be a double point of the conic (i). Hence:

The polar conic of a point on the Hessian of a given cubic is a line-pair meeting at the conjugate point of the Hessian considered as the Jacobian of the polar conics of the given cubic.

§ 3. Ruler-construction for Cubics.

Consider a given cubic as the Jacobian of a family of conics. Denote conjugate pairs of points on the cubic by P and P', Q and Q', ...

If P and P', Q and Q' are conjugate pairs of points on a cubic, so are the intersections of PQ and $P'Q'$, PQ' and $P'Q$.

For P and P' are conjugate for any conic of the family, and so are Q and Q'. But when two pairs of opposite vertices of a quadrilateral are conjugate with respect to a conic, so is the third pair of vertices.* Hence R and R' in Fig. 1 are conjugate for any conic of the family, i.e. are conjugate points on the cubic.

A cubic is uniquely given when three pairs of conjugate points are given.

Expressing the fact that the conic

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

has the three pairs of points as conjugate pairs, we get three linear relations between a, b, c, f, g, h and thus reduce the conic to the form $\lambda U+\mu V+\nu W=0$, where λ, μ, ν are constants. The Jacobian of this family is the cubic required.

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* Hesse's Theorem. It may be proved by projecting the conic into a circle and one vertex into the centre.

Suppose that A and A', B and B', C and C' are the three conjugate pairs of points, then the intersection D of AB and $A'B'$ * and the intersection D' of AB' and $A'B$ are a fourth conjugate pair. From C and C', D and D' we obtain similarly a fifth conjugate pair; and so on. Hence:

Given three pairs of conjugate points on a cubic, we can in general construct an infinite number of conjugate pairs by ruler only.†

Suppose that in Fig. 1 Q and R are consecutive points on the cubic, then the conjugate points Q' and R' are also consecutive points on the curve. We obtain then the result:

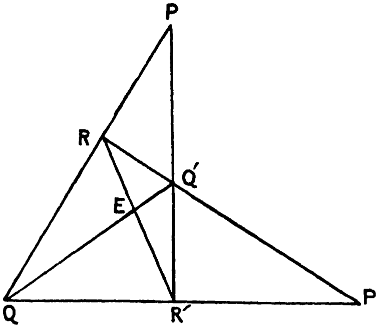

Fig. 1.

The tangents at conjugate points Q and Q' of a cubic meet at a point P on the curve, and QQ' meets the curve again at the point P' conjugate to P.

If, therefore, we draw the four tangents from any point P of a cubic (Fig. 2), the points of contact are two conjugate pairs Q and Q', S and S'; and the lines QQ', SS' meet at P'.

Since the four points of contact can be grouped into two pairs in three different ways, we see that

The points of a given cubic can be divided into conjugate pairs in three different ways.

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* We have tacitly assumed that D does not coincide with C or C'.

† We do not necessarily obtain the *whole* cubic in this way. For special positions of the three given pairs the number of conjugate pairs obtained by the construction may be finite. See Ch. XVI, § 6, Ex. 9.

This also follows from the fact that the cubic is the Hessian of three different cubics, and is therefore the Jacobian of three different families of conics.

By § 2 the polar conic of P with respect to a cubic for which the given cubic is the Hessian is a line-pair meeting at P'. Also by § 1 this line-pair is harmonically conjugate with respect to $P'Q$, $P'Q'$ and with respect to $P'S$, $P'S'$. This shows that the line-pair is QQ', SS'.

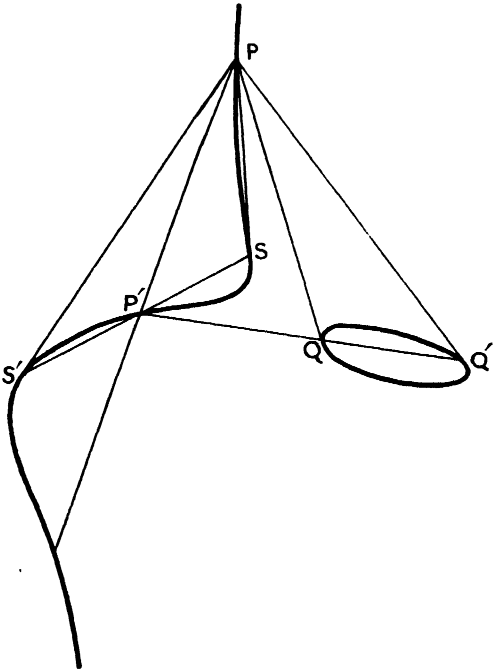

Fig. 2.

The tangent at P' is the harmonic conjugate of PP' with respect to the line-pair QQ', SS'.

For if p is a point on the cubic close to P, $P'(pp', QS)$ is harmonic; and $P'p'$ is the tangent at P' in the limit.

Ex. 1. The *real* points of a cubic with one circuit can be divided into conjugate pairs in only one way. The real points of a cubic with two circuits can be divided into conjugate pairs in three ways. In one way both points of the pair lie on the same circuit. In the other two ways one point lies on each circuit.

[See Ch. XIV, § 1.]

Ex. 2. If three polar conics of a cubic are rectangular hyperbolas, every polar conic is a rectangular hyperbola; and the Hessian is a circular cubic whose singular focus lies on the Hessian.

[The circular points are conjugate points on the Hessian. See also Ch. VII, § 3, Ex. 10.]

Ex. 3. Given three pairs of points A and A', B and B', C and C'; the locus of P such that $P(AA', BB', CC')$ is an involution pencil is a cubic having the given pairs as conjugate pairs of points.

[The cubic is the Jacobian of conics having the given pairs as conjugate pairs. The double rays of the involution are the line-pair of the family of conics through P.]

Ex. 4. A quadrangle is inscribed in a cubic so that its diagonal points lie on the curve. Show that, if we take the vertices as $(1, \pm 1, \pm 1)$, the cubic has the equation

$$ax(y^2-z^2)+by(z^2-x^2)+cz(x^2-y^2)=0.$$

[The quadrangle is $QQ'SS'$ in Fig. 2.]

Ex. 5. Show that in Ex. 4 the tangents at the vertices of the quadrangle meet on the curve at (a, b, c), and that the tangents at

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (a, b, c)$$

meet on the curve at $(1/a, 1/b, 1/c)$.

Ex. 6. Show that the cross-ratios of the pencil of tangents from any point of the cubic of Ex. 4 are $(c^2-a^2)\div(c^2-b^2)$, &c.

[See Ch. VII, § 5, Ex. 2 (vi).]

Ex. 7. If the tangents at four points of a cubic meet on the curve, the polar conic of any one has the triangle formed by the other three as a self-conjugate triangle.

[Consider the polar conic of (a, b, c).]

Ex. 8. Any transversal through a fixed point O of a cubic meets the curve again at P and Q. Show that the locus of the intersection of the tangents at P and Q is a trinodal quartic.

[The locus has evidently nodes at the points conjugate to O in the three ways of dividing the points of the cubic into conjugate pairs. It also touches the cubic at the tangential of O and cuts it at the points of contact of the tangents from O.]

Ex. 9. Any two conjugate points P and P' on the Hessian of Ch. XIII, § 4 (ii) of the cubic of Ch. XIII, § 4 (i) are

$$(\cos\alpha, \sin\alpha, -3\sin 3\alpha) \quad \text{and} \quad (\cos\beta, \sin\beta, -3\sin 3\beta),$$

where $\beta=\alpha+\frac{1}{3}\pi$.

The tangents from P to the Hessian touch at two points collinear with P', and the polar conic of P degenerates into a line-pair through P'.

For the cubic of Ch. XIII, § 4 (iii) and its Hessian we replace sin and cos by sinh and cosh, while $\beta=\alpha+\frac{1}{3}\pi i$.

§ 4. Cayleyan of Cubic.

Suppose now that P, P' are conjugate points on the Hessian of a given cubic (Fig. 2). The envelope of the line PP' is called the *Cayleyan* of the given cubic. It may be considered as the envelope of the line joining any point P on the Hessian to the centre of the degenerate polar conic of P with respect to the given cubic.*

The Cayleyan of a cubic is of the third class. For in Fig. 2 the tangents from P' to the Cayleyan are PP', QQ', SS'.

Each of the lines of the degenerate polar conic of any point on the Hessian of a cubic touches the Cayleyan.

For the polar conic of P is the line-pair QQ', SS' by § 3.

Any two conjugate points Q, Q' on the Hessian of a cubic are harmonic conjugates with respect to the point of contact of QQ' with the Cayleyan and the remaining intersection of QQ' with the Hessian.

For in Fig. 1 suppose R, R' the pair of conjugate points on the Hessian consecutive to Q, Q'. Then P' is the third intersection of QQ' with the Hessian, and the intersection E of QQ', RR' is the point of contact of QQ' with its envelope. But by the harmonic properties of the quadrilateral $(P'E, QQ')$ is a harmonic range in the limit.

§ 5. Tangential Equation of Cayleyan of Cubic.

The tangential equation of the Cayleyan of

$$x^3+y^3+z^3+6mxyz=0$$

is

$$\lambda^3+\mu^3+\nu^3+\frac{1}{m}(1-4m^3)\lambda\mu\nu=0.$$

We proved in § 4 that the Cayleyan touches the lines into which the polar conic of a point (ξ, η, ζ) on the Hessian degenerates. This polar conic is

$$\xi(x^2+2myz)+\eta(y^2+2mzx)+\zeta(z^2+2mxy)=0.$$

If one of the lines into which this conic degenerates is

$$\lambda x+\mu y+\nu z=0,$$

the conic is

$$(\lambda x+\mu y+\nu z)(\xi x/\lambda+\eta y/\mu+\zeta z/\nu)=0.$$

Comparing the coefficients of yz, zx, xy, we have

$$2m\xi\mu\nu-\eta\nu^2-\zeta\mu^2=0$$

and two similar equations.

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* We see that the Steinerian and Hessian of a cubic coincide. See Ch. VII, §§ 8, 9.

Eliminating ξ, η, ζ we get

$$\lambda^3+\mu^3+\nu^3+\frac{1}{m}(1-4m^3)\lambda\mu\nu = 0,$$

which is the tangential equation of the Cayleyan.

Exactly similarly we show that

The tangential equation of the Cayleyan of

$$(x+y+z)^3+6kxyz = 0$$

is $$(\mu+\nu)(\nu+\lambda)(\lambda+\mu) = 2(k+4)\lambda\mu\nu.$$

Ex. 1. The Cayleyan touches the inflexional tangents of a cubic.

[The polar conics of the inflexions touch the Cayleyan. Or use the tangential equation of the Cayleyan.]

Ex. 2. The envelope of a line divided in involution by three given conics is of the third class.

[The Cayleyan of a cubic having the given conics as polar conics. The result may also be obtained by considering the tangents to the locus from any intersection of two of the conics.

For a particular case of the reciprocal of this theorem see § 3, Ex. 3.]

Ex. 3. The poles of any line with respect to a cubic form the vertices of a quadrangle whose sides touch the Cayleyan and whose diagonal points lie on the Hessian.

[The polar conic of any point on the line goes through the four poles by Ch. VII, § 2, Ex. 5.]

Ex 4. The Cayleyan and Hessian touch at the point conjugate to any inflexion P of the Hessian.

[Make P consecutive to an inflexion in Fig. 2. Or verify that the two curves touch at the points $(m, m, 1)$, &c.]

Ex. 5. The Cayleyan of a non singular cubic is a curve of the sixth degree with nine cusps.

The cuspidal tangents are the harmonic polars of the inflexions of the given cubic.

[The cuspidal tangents and cusps of the Cayleyan are obtained in the same way as the inflexions and inflexional tangents of

$$x^3+y^3+z^3+6mxyz = 0.]$$

Ex. 6. If A and A' are conjugate points on the Hessian, the polar of A with respect to the polar conic of A' for any syzygetic cubic touches the Cayleyan.

Ex. 7. The locus of a point whose polar line with respect to the cubic touches the Cayleyan is a sextic.

Ex. 8. Discuss the case $m = 0$ in § 5.

Ex. 9. Reciprocate the mutual relationships of cubic, Hessian, and Cayleyan.

Ex. 10. Find the tangential equation of the Cayleyan of

$$\text{(i) } y^2z = 4x^3 - g_2xz^2 - g_3z^3, \qquad \text{(ii) } z^2x = y(y-x)(y-k^2x),$$
$$\text{(iii) } ax(y^2-z^2) + by(z^2-x^2) + cz(x^2-y^2) = 0.$$

[Equating the coefficients of x^2, y^2, z^2, yz, zx, xy in

$$(\lambda x + \mu y + \nu z)(\lambda' x + \mu' y + \nu' z) \qquad \text{and} \qquad \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z},$$

and eliminating ξ, η, ζ, λ', μ', ν' between the six linear equations thus obtained, we have the tangential equation of the Cayleyan of the cubic $f = 0$.]

Ex. 11. The Cayleyan of a cuspidal cubic degenerates into the cusp and the intersection of the cuspidal and inflexional tangents.

Ex. 12. The Cayleyan of a nodal cubic is a conic having as the vertices of a self-conjugate triangle the node A, any inflexion C, and the intersection B of the harmonic polar of C with the line of inflexions.

[The Cayleyan of $z(x^2 \pm y^2) = y(3x^2 \mp y^2)$ is $z^2 = 9(y^2 \pm x^2)$. See § 3, Ex. 9, for the coordinates of a pair of corresponding points on the Hessian; or use the method of Ex. 10.]

CHAPTER XVI

USE OF PARAMETER FOR NON-SINGULAR CUBICS

§ 1. A Standard Equation of the Cubic with two Circuits.

The equation of any cubic with two circuits can be put in the form

$$z^2x = y(y-x)(y-k^2x), \text{ where } 1 > k^2 > 0,$$

by a suitable choice of homogeneous coordinates.

Take a triangle of reference ABC such that BC is the tangent at a real inflexion C, and AB is the harmonic polar of C.

Since the equation of the cubic must reduce to $y^3 = 0$ when we put $x = 0$ in this equation, the coefficients of z^3, y^2z, yz^2 in the equation are zero.

Since the polar conic of $(0, 0, 1)$ is $zx = 0$, the coefficients of x^2z and xyz are also zero. Choosing homogeneous coordinates such that the coefficients of z^2x and y^3 are equal and opposite, the equation becomes

$$z^2x-(y-\alpha x)(y-\beta x)(y-\gamma x) = 0.$$

The points $(1, \alpha, 0)$, $(1, \beta, 0)$, $(1, \gamma, 0)$ are the points of contact of the tangents to the cubic from $(0, 0, 1)$, and are real since the cubic has two circuits. Suppose $\alpha > \beta > \gamma$.

Put now

$$x = \frac{X}{\alpha-\gamma}, \quad y = Y + \frac{\gamma X}{\alpha-\gamma}, \quad z = (\alpha-\gamma)^{\frac{1}{2}} Z, \quad \frac{\beta-\gamma}{\alpha-\gamma} = k^2,$$

and the equation becomes

$$Z^2X = Y(Y-X)(Y-k^2X).$$

Replacing X, Y, Z by x, y, z we have the required form.

The point $(1, 0, 0)$ is now the point of contact of a tangent from C.

§ 2. Coordinates of any Point on a Cubic with two Circuits.

Any point on the cubic

$$z^2x = y(y-x)(y-k^2x)$$

may be taken as

$$(\operatorname{sn}^3 u, \ \operatorname{sn} u, \ \operatorname{cn} u \operatorname{dn} u),$$

the modulus of the elliptic functions being k, as is at once verified.

Remembering that,* if m and n are integers,

$$\begin{aligned} \operatorname{sn}(u+2mK+2nK'i) &= (-1)^m \operatorname{sn} u, \\ \operatorname{cn}(u+2mK+2nK'i) &= (-1)^{m+n} \operatorname{cn} u, \\ \operatorname{dn}(u+2mK+2nK'i) &= (-1)^n \operatorname{dn} u, \end{aligned}$$

we see that the point on the cubic is unaltered, if we replace u by $u+2mK+2nK'i$.

The points with parameters u_1, u_2, u_3 *are collinear if*

$$u_1+u_2+u_3 \equiv 0 \ (\text{mod. } 2K,\ 2K'i).$$

This is equivalent to $\operatorname{sn}(u_1+u_2+u_3) = 0$. Denote $\operatorname{sn} u_1$, $\operatorname{cn} u_1$, $\operatorname{dn} u_1$, ... by s_1, c_1, d_1, ... Then $\operatorname{cn}(u_1+u_2+u_3) = \pm 1$ gives

$$\pm c_3 = \operatorname{cn}(u_1+u_2) = (s_1 c_1 d_2 - s_2 c_2 d_1) \div (s_1 c_2 d_2 - s_2 c_1 d_1).*$$

Hence

$$\frac{d_1}{s_1(c_2 c_3 \mp c_1)} = \frac{d_2}{s_2(c_3 c_1 \mp c_2)} = \frac{d_3}{s_3(c_1 c_2 \mp c_3)}.$$

Substituting for $d_1 : d_2 : d_3$ in the determinant

$$\begin{vmatrix} s_1^3 & s_1 & c_1 d_1 \\ s_2^3 & s_2 & c_2 d_2 \\ s_3^3 & s_3 & c_3 d_3 \end{vmatrix},$$

we find that it vanishes. But the vanishing of this determinant is the condition that the points with parameters u_1, u_2, u_3 should be collinear.

Another line of argument is to employ the formula

$$\operatorname{sn}(u_1+u_2+u_3) = \begin{vmatrix} s_1^3 & s_1 & c_1 d_1 \\ s_2^3 & s_2 & c_2 d_2 \\ s_3^3 & s_3 & c_3 d_3 \end{vmatrix} \div \begin{vmatrix} 1 & s_1^2 & c_1 d_1 \\ 1 & s_2^2 & c_2 d_2 \\ 1 & s_3^2 & c_3 d_3 \end{vmatrix}.$$

To obtain the inflexions take u_1, u_2, u_3 all equal to u. We have then $\operatorname{sn} 3u = 0$ or

$$u = \tfrac{2}{3}\epsilon K + \tfrac{2}{3}\epsilon' K' i \quad (\epsilon,\ \epsilon' = 0,\ 1,\ 2).$$

The real inflexions are given by

$$u = 0, \quad u = \tfrac{2}{3}K, \quad u = \tfrac{4}{3}K.$$

The odd circuit of the curve is given by real values of u lying between 0 and $2K$.

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* See A. C. Dixon's *Elliptic Functions*, §§ 22, 38.

The even circuit of the curve is given by $u = v + K'i$, where v is real and lies between 0 and $2K$.*

Ex. 1. Use § 2 to prove that the inflexions lie by threes on twelve straight lines.

Ex. 2. The four tangents from any point of the cubic to the curve form a pencil, one of whose cross-ratios is k^2.

[Consider the tangents from the inflexion C.]

Ex. 3. The points of contact of the four tangents from the point on the curve with parameter u are

$$(s^3, s, cd), \quad (c^3, cd^2, -k'^2 sd), \quad (1, k^2 s^2, -k^2 scd), \quad (d^3, k^2 c^2 d, k^2 k'^2 sc);$$

where s, c, d denote

$$\operatorname{sn}(-\tfrac{1}{2}u), \quad \operatorname{cn}(-\tfrac{1}{2}u), \quad \operatorname{dn}(-\tfrac{1}{2}u), \quad \text{and} \quad k^2 + k'^2 = 1.$$

[The parameters of the points of contact are

$$-\tfrac{1}{2}u, \quad -\tfrac{1}{2}u + K, \quad -\tfrac{1}{2}u + K'i, \quad -\tfrac{1}{2}u + K + K'i.]$$

Ex. 4. Obtain the cross-ratio of the range in which the tangents at the points with parameters

$$v, \quad v + K, \quad v + K'i, \quad v + K + K'i$$

meet a side of the triangle of reference, and verify that the cross-ratio of the pencil formed by the tangents from a point on a cubic is constant.

[The method of § 7, Ex. 1, is also available.]

§ 3. Parameters of Points on a Conic or Cubic.

If $u_1, u_2, u_3, u_4, u_5, u_6$ *are the parameters of points on a conic,*

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \equiv 0.$$

The congruence is taken modulo $2K$, $2K'i$; and so always unless the contrary is stated.

For suppose the three lines joining the points with parameters u_1 and u_2, u_3 and u_4, u_5 and u_6 meet the cubic again in points with parameters v_{12}, v_{34}, v_{56}. Then

$$u_1 + u_2 + v_{12} \equiv 0, \quad u_3 + u_4 + v_{34} \equiv 0, \quad u_5 + u_6 + v_{56} \equiv 0.$$

But the given cubic, the three lines just mentioned, the conic and the line through the points with parameters v_{12} and v_{34} are three cubics through the points with parameters

$$u_1, u_2, u_3, u_4, u_5, u_6, v_{12}, v_{34}.$$

They therefore all pass through the point with parameter v_{56} (Ch. XII, § 2). Hence the points with parameters v_{12}, v_{34}, v_{56} are collinear, and $v_{12} + v_{34} + v_{56} \equiv 0$. Hence

$$u_1 + u_2 + u_3 + u_4 + u_5 + u_6 \equiv 0.$$

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* For $\operatorname{sn}(v + K'i) = 1/k \operatorname{sn} v$, $\operatorname{cn}(v + K'i) = i \operatorname{dn} v/k \operatorname{sn} v$, $\operatorname{dn}(v + K'i) = i \operatorname{cn} v/\operatorname{sn} v$, see A. C. Dixon's *Elliptic Functions*, § 18.

If $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ *are the parameters of the intersections of the given cubic with another cubic,*

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9 \equiv 0.$$

Suppose that the conic through the points with parameters u_1, u_2, u_3, u_4, u_5 meets the cubic again in a point with parameter p; and that the two lines joining the points with parameters u_6 and u_7, u_8 and u_9 meet the curve again in points with parameters q, r. Then

$$u_1+u_2+u_3+u_4+u_5+p \equiv 0, \quad u_6+u_7+q \equiv 0, \quad u_8+u_9+r \equiv 0.$$

Now of the twelve intersections of the given cubic and the quartic consisting of the conic and the two lines just mentioned, nine lie on another cubic. Therefore the remaining three lie on a straight line.*

Hence $p+q+r \equiv 0$; and therefore

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8+u_9 \equiv 0.$$

§ 4. A Standard Equation of the Cubic with one Circuit.

The equation of any cubic with one circuit can be put in the form

$$z^2(y-x) = (y+x)(k^2x^2+k'^2y^2); \textit{ where } k^2+k'^2 = 1.$$

As in § 1, take C as a real inflexion and AB as the harmonic polar of C. But take A, B as the double points of the involution determined by (1) the intersection of AB with the tangent at C and the real intersection of AB with the cubic, (2) the pair of unreal intersections of AB with the cubic. Choose also the homogeneous coordinates so that the tangent at C is $x = y$. Then, as in § 1, the equation of the cubic contains no term in z^3, x^2z, xyz, or y^2z, and takes the form

$$z^2(y-x) = (y+x)(ax^2+by^2).$$

In this a and b have the same sign. There is no loss of generality in supposing them both positive, for we can interchange x and y if they are negative. Now replace z by $(a+b)^{\frac{1}{2}}z$, and the equation of the cubic is in the required form.

Any point on this curve is

$$(\operatorname{sn} u \ \operatorname{cn} u, \quad \operatorname{sn} u, \quad (1+\operatorname{cn} u)\operatorname{dn} u),$$

the modulus of the elliptic functions being k. The point on the cubic is unaltered, if we replace u by

$$u+2m(K+K'i)+2n(K-K'i).$$

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* Ch. XII, § 7, Ex 4. More rigorous is the proof by Abel's theorem, Ch. X, § 7, Ex. 7.

As in § 2 the parameters u_1, u_2, u_3 of three collinear points are connected by the relation

$$u_1+u_2+u_3 \equiv 0,$$

the congruence being now taken modulo $2(K+K'i)$ and $2(K-K'i)$. The proof is as in § 2, noting that for three collinear points $\operatorname{cn}(u_1+u_2+u_3) = +1$.

The inflexions have parameters

$$\tfrac{2}{3}\epsilon(K+K'i)+\tfrac{2}{3}\epsilon'(K-K'i), \quad (\epsilon, \epsilon' = 0, 1, 2).$$

The real inflexions have parameters $0, \frac{4}{3}K, \frac{8}{3}K$. The real part of the curve is given by real values of u lying between 0 and $4K$.

The connexions between the parameters of the points in which the curve meets a conic or another cubic are the same as in the case of the cubic with two circuits, except that the congruences are taken modulo $2(K+K'i)$ and $2(K-K'i)$.

Ex. Show that the equation of a cubic with two circuits can be put in the form

$$z^2(y-x) = (y+x)(y^2-k^2x^2), \quad \text{where} \quad 1 > k^2 > 0;$$

and express the coordinates of any point rationally in terms of elliptic functions.

[C is the inflexion, and A and B are the double points of the involution determined by the intersections of the harmonic polar of C (1) with the odd circuit and the tangent at C, (2) with the even circuit.]

§ 5. Unicursal Cubic.

If (x, y, z) is a double point of

$$z^2x = y(y-x)(y-k^2x),$$

then (Ch. II, § 4)

$$z^2+(1+k^2)y^2-2k^2xy = -3y^2+2(1+k^2)xy-k^2x^2 = zx = 0.$$

Rejecting the case $x = y = z = 0$, we get

either $k = 0$, $y = z = 0$,

when the cubic becomes

$$z^2x = y^2(y-x),$$

with an acnode at $(1, 0, 0)$;

or $k = 1$, $x-y = z = 0$,

when the cubic becomes

$$z^2x = y(y-x)^2,$$

with a crunode at $(1, 1, 0)$.

Similarly (x, y, z) is a double point of

$$z^2(y-x) = (y+x)(k^2x^2+k'^2y^2), \quad \text{where} \quad k^2+k'^2 = 1,$$

either if $k = 0$, $y = z = 0$, when the cubic becomes

$$z^2(y-x) = y^2(y+x),$$

with an acnode at $(1, 0, 0)$;

or if $k = 1$, $x = z = 0$, when the cubic becomes

$$z^2(y-x) = x^2(y+x)$$

with a crunode at $(0, 1, 0)$.

If the cubic has a double point, the elliptic functions degenerate into trigonometrical functions, as could have been foreseen, since the cubic is now unicursal.

We see that the standard equation of a cubic found in either § 1 or § 4 includes the case of an acnodal or crunodal cubic, but not that of a cuspidal cubic.

In Ch. XIII we expressed the coordinates of any point on a *unicursal* cubic in terms of a parameter in such a way that the sum of the parameters of three collinear points $\equiv 0$. Hence many of the results established for non-singular cubics hold good with slight modification for unicursal cubics. We shall leave the reader in general to find out for himself what modification is necessary in any particular case.

§ 6. Applications of the Parameter.

We now apply the parametric representation of a point on a cubic to the investigation of various properties of the curve.

A cubic has twenty-seven sextactic points, namely, the points of contact of tangents from the inflexions.

A *sextactic* point of a curve is one at which the conic of closest contact has six-point contact.

If u is the parameter of such a point, we must have $6u \equiv 0$, for the sum of the parameters of six points on a conic $\equiv 0$.

This gives

$$u = \frac{\epsilon}{6}M + \frac{\epsilon'}{6}N, \quad (\epsilon, \epsilon' = 0, 1, 2, 3, 4, 5);$$

where $$M = 2K, \quad N = 2K'i$$

for the cubic with two circuits, and

$$M = 2(K+K'i), \quad N = 2(K-K'i)$$

for the cubic with one circuit.

But this formula includes the inflexions (the conic being the inflexional tangent taken twice), namely, when ϵ and ϵ' are both even. There remain twenty-seven values of u giving sextactic points.

If the tangential of such a point has the parameter v,

$$u+u+v \equiv 0, \text{ or } v \equiv -2u = -\frac{\epsilon}{3}M-\frac{\epsilon'}{3}N.$$

Hence the tangential of a sextactic point is an inflexion.

A cubic has seventy-two coincidence points, namely, the points coinciding with their third tangential.

A *coincidence* point of a curve is one at which an infinite number of cubics can be drawn having nine-point contact with the curve.

If u is the parameter of such a point, we must have $9u \equiv 0$, and therefore

$$u = \frac{\epsilon}{9}M+\frac{\epsilon'}{9}N, \quad (\epsilon, \epsilon' = 0, 1, 2, 3, 4, 5, 6, 7, 8).$$

Excluding the case in which ϵ and ϵ' are multiples of 3, which gives the inflexions, we have seventy-two coincidence points.

As shown above, the tangential of the point with parameter u has parameter $-2u$, the second tangential (tangential of the tangential) has parameter $-2(-2u) = 4u$, and the third tangential has parameter $-2 \cdot 4u = -8u$. Therefore a point coincides with its third tangential if $u \equiv -8u$ or $9u \equiv 0$.

If the tangents at the points with parameters u, v meet on the curve, i.e. the points have the same tangential, $2u \equiv 2v$ and therefore, since $v \not\equiv u$,

$$v = u+\tfrac{1}{2}M, \; u+\tfrac{1}{2}N, \text{ or } u+\tfrac{1}{2}M+\tfrac{1}{2}N.$$

Hence when the points of a cubic are grouped into conjugate pairs (Ch. XV, § 3), the parameters of a pair differ by $\frac{1}{2}M$, $\frac{1}{2}N$, or $\frac{1}{2}(M+N)$. It will be remembered that there are three different ways of grouping the points into conjugate pairs.

If, however, we confine ourselves to the real points of the cubic, we have only one method of pairing the points of the cubic with one circuit, and three methods of pairing the points of the cubic with two circuits.

Ex. 1. If one polygon of $2n$ sides can be inscribed in a given cubic with the sides passing alternately through P and Q, where P and Q are given points of the cubic, an infinite number of such polygons can be inscribed.

[Suppose $u_1, u_2, \ldots, u_{2n}$ are the parameters of the vertices of such a polygon, and p, q are the parameters of P, Q. Then we must have the congruences,

$$p+u_1+u_2 \equiv 0, \quad p+u_3+u_4 \equiv 0, \quad \ldots, \quad p+u_{2n-1}+u_{2n} \equiv 0;$$
$$q+u_2+u_3 \equiv 0, \quad q+u_4+u_5 \equiv 0, \quad \ldots, \quad q+u_{2n}+u_1 \equiv 0;$$

which are consistent if $n(p-q) \equiv 0$.

P and Q are called *Steiner's points*. See Clebsch, *Crelle*, lxiii (1864), p. 106.]

Ex. 2. Show that for the polygons of Ex. 1

(i) If $n=2$, the tangents at P and Q meet on the curve; so that P and Q are conjugate points on the cubic.

(ii) If $n=3$, PQ_1 and QP_1 meet on the curve; where P_1 and Q_1 are the tangentials of P and Q.

Any two inflexions form a Steiner pair for inscribed hexagons.

(iii) If $n=5$, PQ_3 and QP_3 meet on the curve; where PQ_1 and QP_1 meet the curve again in P_2 and Q_2, and PQ_2 and QP_2 meet the curve again in P_3 and Q_3.

The conics of closest contact at P and Q meet on the cubic.

(iv) Any two sextactic points form a Steiner pair for inscribed dodecagons.

Ex. 3. The two points in which the lines joining any point of the cubic to a pair of Steiner points meet the curve again are also a Steiner pair.

Ex. 4. If P, Q and P', Q' are two Steiner pairs, the lines PP' and QQ' meet the curve again in a Steiner pair.

Ex. 5. If P, Q is a Steiner pair for a polygon of $2n$ sides, and PQ meets the curve in the tangential of R, then P, R (and Q, R) is a Steiner pair for a polygon of $4n$ sides.

Ex. 6. If all but one of the $3n$ points, which are the vertices of a polygon of $2n$ sides and the intersections of opposite sides of the polygon, lie on a cubic, so does the remaining point.

[If $u_1, u_2, \ldots, u_{2n}$ are the parameters of the vertices, we have to prove

$$u_1+u_2 \equiv u_n+u_{n+1}, \quad u_2+u_3 \equiv u_{n+1}+u_{n+2}, \quad \ldots$$

consistent; which is evidently the case.

Pascal's theorem is a particular case of this result. The reader may generalize the theorem. For instance, consider the case

$$u_1+u_2 \equiv u_4+u_5, \quad u_3+u_4 \equiv u_6+u_7, \quad \ldots,$$
$$u_{2n-3}+u_{2n-2} \equiv u_{2n}+u_1, \quad u_{2n-1}+u_{2n} \equiv u_2+u_3.]$$

Ex. 7. If $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$ are given points on a cubic such that one polygon of $2n$ sides can be inscribed in the cubic with its sides passing in order through $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$, then an infinite number of polygons can be thus inscribed.

Ex. 8. Three triangles can be inscribed in a given cubic whose sides meet the curve again in three given collinear points.

Ex. 9. Three non-collinear points are taken on a cubic with parameters u, v, w. The lines joining any two of these meet the cubic again in three more points. The line joining two of the six points now obtained meets the cubic again in a fresh point, and so on. Show that the points thus obtained have parameters given by the expression $mu+nv+pw$, where m, n, p are positive or negative integers such that $m+n+p-1$ is a multiple of 3.

Under what conditions is the total number of points obtained finite? *

[The points with parameters

$-v-w$, $-w-u$, $-u-v$, $(v+w)+(w+u)$, $u+v-(u+v+2w)=-2w$,

and similarly $-2u$, $-2v$ are among those obtained. Now assume the result true for all values of m, n, p numerically less than those considered, and use induction.

The number of points is finite if u, v, w are commensurable with any periods.]

Ex. 10. The tangents at A, B, C, D on a cubic meet on the curve, and P is any other point of the cubic. The lines PA, PB, PC, PD meet the curve again at A', B', C', D'. Show that the tangents at A', B', C', D' meet on the curve, and that the line joining A, B, C, or D to A', B', C', or D' passes through one of four points P, Q, R, S on the curve, such that the tangents at P, Q, R, S meet on the curve.

Ex. 11. An infinite number of triangles can be inscribed in a cubic such that the tangent at each vertex meets the opposite side on the curve. Any side is divided harmonically by the cubic and its point of contact with its envelope.

[If u, v, w are the parameters of the vertices,

$$3u \equiv 3v \equiv 3w \equiv u+v+w.$$

See also Ch. XII, § 4, Ex. 21.]

Ex. 12. Find the number of conics through four given points of a cubic which touch the curve, and the number of conics through three given points of a cubic which osculate the curve.

[Easy also by quadratic transformation.]

Ex. 13. Nine or three of the sextactic points are real. Three of them lie on the odd circuit, and six on the even circuit, if it exists.

Ex. 14. Six of the coincidence points are real. They all lie on the odd circuit.

Ex. 15. The coincidence points are the vertices of the 24 'Hart triangles', both inscribed and circumscribed to the cubic.

Ex. 16. If a conic has five-point contact with a cubic, it meets the cubic again on the line joining the point of contact to its second tangential.

Deduce the fact that the sextactic points are the points of contact of tangents from the inflexions.

[$5u+v\equiv 0$, $u+4u+w\equiv 0$ give $v\equiv w$.]

Ex. 17. If a conic has four-point contact with a cubic, the line joining the other two intersections of the conic and cubic passes through the second tangential of the point of contact.

[$4u+v+w\equiv 0$, $p+v+w\equiv 0$ give $p\equiv 4u$.]

Ex. 18. All cubics having eight-point contact with a given cubic at a given point pass through the third tangential of the point of contact.

Deduce the fact that a coincidence point coincides with its third tangential.

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* If not, the construction gives the whole of the odd circuit or the whole of both circuits. See Hurwitz, *Crelle*, cvii, p. 141.

Ex. 19. The locus of the sextactic points of a pencil of syzygetic cubics is nine straight lines.

[The harmonic polars of the inflexions. The locus of the coincidence points is eight equianharmonic cubics; see Halphen, *Math. Annalen*, xv (1879), p. 359. We may get their equation by identifying P with its third tangential in Ch. XIV, § 9, Ex. 3.]

Ex. 20. A cuspidal cubic has no sextactic or coincidence points.

Ex. 21. A nodal cubic has three sextactic and six coincidence points. How many are real?

Ex. 22. From the number of sextactic points of a cubic, deduce that any curve has $12m-15n+9\kappa$ sextactic points.

[We assume that the number is the same for all curves of the same type. Let it be $\phi(n, m, \alpha)$, where α is $3m+\kappa$. Now if $f=0$, $f_1=0$ are the equations of two curves, $ff_1=\epsilon$ is a curve very close to the degenerate curve $ff_1=0$, ϵ being a small constant. Hence for all values of $n, m, \alpha, n', m', \alpha'$ we must have

$$\phi(n, m, \alpha)+\phi(n', m', \alpha') = \phi(n+n', m+m', \alpha+\alpha').$$

The theory of functional equations gives readily that $\phi = an+bm+c\alpha$, where a, b, c are constants not involving n, m, α. Now a, b, c are at once given by noting that

$$n=3,\ m=6,\ \alpha=18,\ \phi=27;\quad n=3,\ m=4,\ \alpha=12,\ \phi=3;$$
$$n=3,\ m=3,\ \alpha=10,\ \phi=0$$

must satisfy the relation $\phi=an+bm+c\alpha$.]

Ex. 23. The number of coincidence points on a curve is

$$33m-42n+27\kappa.$$

[As in Ex. 22. For a more general result see Halphen, *Bull. de la Soc. Math. de France*, iv (1876), pp. 59-85.]

Ex. 24. Three families of conics can be drawn touching a given cubic at three distinct points. If a conic is drawn through the three points of contact of any such conic, it meets the cubic again in the points of contact of a triply-tangent conic of the same family.

Ex. 25. If six points of a cubic lie on a conic, so do the six conjugate points.

[If $u_1+u_2+\ldots+u_6\equiv 0$, $(u_1+\tfrac{1}{2}M)+(u_2+\tfrac{1}{2}M)+\ldots+(u_6+\tfrac{1}{2}M)\equiv 0$]

Ex. 26. If a second cubic passes through nine points of a given cubic, another cubic passes through one of the points and the points conjugate to the other eight.

§ 7. Another Standard Equation of the Cubic

The equation of any cubic can be put in the form

$$y^2z = 4x^3-g_2xz^2-g_3z^3$$

by a suitable choice of homogeneous coordinates.

In this equation g_2 and g_3 are any constants, the notation being that which is usual in the theory of elliptic functions.

Take as sides of the triangle of reference the tangent at a real inflexion B, the harmonic polar of B, and the polar line of the intersection A of this tangent and harmonic polar.

Since the equation of the cubic must reduce to $x^3 = 0$ when we put $z = 0$ in this equation, the coefficients of x^2y, xy^2, and y^3 in the equation are zero.

Since the polar conic of (0, 1, 0) is $yz = 0$, the coefficients of xyz and yz^2 are zero.

Since the polar line of (1, 0, 0) is $x = 0$, the coefficient of x^2z is zero.

We see then that the cubic is of the form

$$ax^3 + by^2z + cxz^2 + dz^3 = 0.$$

Replacing now x by $(-4b/a)^{\frac{1}{3}}x$, we get the required form.

The cubic has one or two circuits according as the harmonic polar $y = 0$ of the real inflexion meets the curve in one or three real points; that is, according as $g_2{}^3 - 27g_3{}^2$ is less than or greater than zero.

The usual condition (Ch. II, § 4) shows that, if (x, y, z) is a double point of the curve,

$$yz = 12x^2 - g_2z^2 = y^2 + 2g_2xz + 3g_3z^2 = 0,$$

leading to $g_2{}^3 = 27g_3{}^2$, and to

$$x : y : z = -g_3{}^{\frac{1}{3}} : 0 : 2.$$

It will be readily found that in this case the double point is a crunode, cusp, or acnode, according as g_3 is $<$, $=$, or >0.

Hence the equation of every (non-degenerate) cubic can be put in the given form.

Ex. 1. Use the equation of a cubic given in § 7 to prove that the pencil of tangents from any point of a cubic has a constant cross-ratio.

[Let $(\xi, \eta, 1)$ be a point P on the curve. The line

$$y - \eta z = m(x - \xi z)$$

through P meets the curve where $x = \xi z$ and where

$$4x^2 + (4\xi - m^2)xz + (m^2\xi - 2m\eta + 4\xi^2 - g_2)z^2 = 0.$$

The line is therefore a tangent if this equation in x/z has equal roots, which gives $$m^4 - 24\xi m^2 + 32\eta m + 16g_2 - 48\xi^2 = 0.$$

Since the line meets $z = 0$ at $(1, m, 0)$, the cross-ratio ϕ of the pencil of the tangents from P is given by (Ch. I, § 11)

$$(\phi+1)^2(\phi-2)^2(\phi-\tfrac{1}{2})^2I^3 = 27(\phi^2-\phi+1)^3J^2,$$

where $$I = 16g_2, \quad J = 64g_3.]$$

Ex. 2. The cross-ratio of the tangents from any point of the curve is harmonic if $g_3 = 0$, and equianharmonic if $g_2 = 0$.

§ 8. Coordinates in terms of Weierstrass's Function.

If the cubic

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3$$

is non-singular, we may take any point on the curve as $(\wp u, \wp' u, 1)$; where $\wp u$ is Weierstrass's elliptic function given by

$$(\wp' u)^2 = 4(\wp u)^3 - g_2\wp u - g_3\,; \quad \mathop{\mathrm{L}^t}_{u\to 0} u^2 . \wp u = 1.$$

If the points with parameters u, v, w are collinear,

$$\begin{vmatrix} \wp u & \wp' u & 1 \\ \wp v & \wp' v & 1 \\ \wp w & \wp' w & 1 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

This gives

$$u + v + w \equiv 0 \pmod{2\omega, 2\omega'},$$

where 2ω, $2\omega'$ are the periods of $\wp u$.

In fact (Dixon's *Elliptic Functions*, § 72), if $u+v+w \equiv 0$,

$$\wp u + \wp v + \wp w = \tfrac{1}{4}\left(\frac{\wp' v - \wp' w}{\wp v - \wp w}\right)^2 = \tfrac{1}{4}\left(\frac{\wp' w - \wp' u}{\wp w - \wp u}\right)^2$$
$$= \tfrac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2.$$

Hence $\pm \dfrac{\wp' v - \wp' w}{\wp v - \wp w} = \pm \dfrac{\wp' w - \wp' u}{\wp w - \wp u} = \pm \dfrac{\wp' u - \wp' v}{\wp u - \wp v}$;

from which (i) immediately follows.*

The inflexions have parameters

$$\tfrac{2}{3}(\epsilon\omega + \epsilon'\omega'), \quad (\epsilon, \epsilon' = 0, 1, 2).$$

The real inflexions have parameters $0, \dfrac{2\omega}{3}, \dfrac{4\omega}{3}$; where 2ω is the real period.

The odd circuit is given by real values of the parameter. If $g_2^3 > 27g_3^2$, the cubic has also an even circuit given by parameters of the form $u+\omega'$, where u is real.

The connexions between the parameters of the points in which the curve meets a conic or another cubic are the same as in the case of the cubic with two circuits in § 3, except that the congruences are taken modulo 2ω. and $2\omega'$. The standard form of this and the preceding section may be used as in § 6 to establish properties of the cubic instead of the standard forms of §§ 1, 4.

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* It is at once proved that all the plus or all the minus signs must be taken.

CHAPTER XVII

UNICURSAL QUARTICS

§ 1. Types of Quartic.

A QUARTIC cannot have more than three double points (Ch. VIII, § 2), so that its deficiency may be 0, 1, 2, or 3. By the aid of Ch. VIII, § 1, we may draw up a table showing all possible types of quartic as follows:

Type	n	m	δ	κ	τ	ι	D
(i)	4	12	0	0	28	24	3
(ii)	4	10	1	0	16	18	2
(iii)	4	9	0	1	10	16	2
(iv)	4	8	2	0	8	12	1
(v)	4	7	1	1	4	10	1
(vi)	4	6	0	2	1	8	1
(vii)	4	6	3	0	4	6	0
(viii)	4	5	2	1	2	4	0
(ix)	4	4	1	2	1	2	0
(x)	4	3	0	3	1	0	0

To these types must be added the quartic with a triple point, or higher singularity.

In this chapter we discuss quartics of zero deficiency, namely types (vii) to (x) of the above table, reserving the discussion of the other types for Chapters XVIII and XIX.

§ 2. Geometrical Methods.

Suppose a quartic has three double points A, B, C. We may convert it by quadratic transformation into a conic, taking A, B, C as the vertices of the triangle in Ch. IX, Fig. 1. We thus deduce the properties of the unicursal quartic from those of the conic.

A process which comes to the same thing is to project B and C into the circular points and then invert with respect to A, when the quartic becomes a conic (a parabola, if A is a cusp); and from each property of the conic a property of the quartic is deduced.

In fact, it is readily seen that the equation of a quartic with double points at the origin and circular points is of the form

$$c\,(x^2+y^2)^2+2\,(gx+fy)\,(x^2+y^2)+ax^2+2hxy+by^2=0,$$

since every circular line meets it in only two finite points (Ch. II, § 5). An inverse of this with respect to the origin is the conic

$$ax^2+2hxy+by^2+2\,gx+2fy+c=0.$$

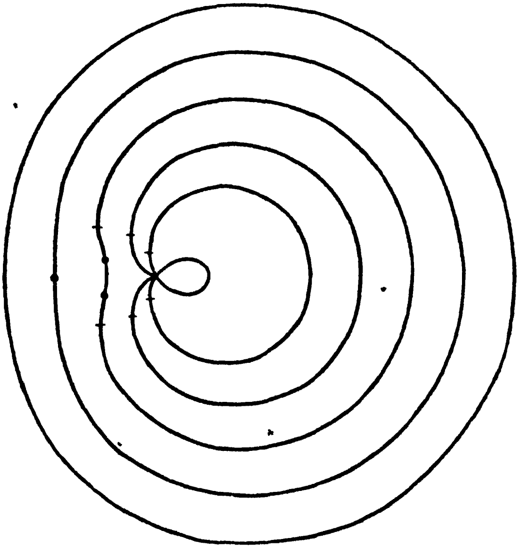

Fig. 1.

The limaçon $r=a+b\cos\theta$ for $a/b=\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.

The method of projection and inversion will, however, be suitable only if B and C are double points of the same kind, e. g. both cusps or both ordinary nodes. It is not readily applicable if all the double points are of different kinds; for instance, a cusp, an ordinary node, and a flecnode. But quadratic transformation is still available.

In the case of a quartic with a node A and two cusps B, C each property of the curve can be duplicated by reciprocation;

for the reciprocal of such a curve is a similar quartic, as is evident from its Plücker's numbers.

If we project B, C into the circular points, and then invert with respect to A, the quartic becomes a conic with A as focus; since the inverse of a curve with respect to a focus has cusps at the circular points, and conversely (see Ch. V, § 4, and especially Ex. 1 to 6 in that section).

Hence, if we project the cusps of a quartic with a node and two cusps into the circular points, it becomes the inverse of a conic with respect to a focus, i.e. a limaçon with polar equation of the form

$$r = a + b\cos\theta.$$

Fig. 1 shows five limaçons with the same b and

$$a/b = \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ 2,\ \tfrac{5}{2}.$$

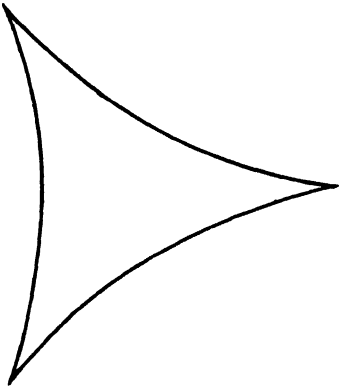

Fig. 2.
Three-cusped hypocycloid.

The properties of a quartic with three cusps may be investigated geometrically in various ways. For instance, we may obtain its properties by reciprocation from the known properties of the nodal cubic.

Or again, if we project two of the cusps into the circular points, the curve becomes a limaçon with a finite cusp, i.e. the cardioid

$$r = a(1 + \cos\theta),$$

which is the inverse of a parabola with respect to its focus.

Or again, if we project the points of contact of the bitangent into the circular points, the curve becomes a three-cusped hypocycloid (Fig. 2).

For the polar reciprocal of the curve with respect to a cusp A will be a cubic touching the circular lines at A and having the line at infinity as inflexional tangent. Its equation can therefore be put in the form

$$c(x^2 + y^2) = x^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

A being the origin.

But the parametric equations of a three-cusped hypocycloid are

$$x = \tfrac{1}{3}a(2\cos\phi + \cos 2\phi),\ y = \tfrac{1}{3}a(2\sin\phi - \sin 2\phi) \ . \ \text{(ii)},$$

the cusps being $\phi = 0,\ \frac{2}{3}\pi,\ \frac{4}{3}\pi$, and the origin the centre of the curve. Writing down the polar of the point (ii) with respect to the circle

$$(x-a)^2+y^2=b^2$$

and finding its envelope, by differentiating with respect to ϕ in the usual manner, we obtain

$$3b^2\{(x-a)^2+y^2\}+4a(x-a)^3=0$$

as the polar reciprocal of the hypocycloid with respect to a cusp. But this is the curve (i).

Another proof is given in § 5.

Ex. 1. A quartic has nodes A, B, C. Through A is drawn any line meeting the quartic again in P, Q and BC in R. Find the locus of S if (PQ, RS) is harmonic.

[Project B and C into the circular points. Then we have: 'Find the locus of the middle point S of a chord PQ of a bicircular quartic meeting the quartic again at its finite node A.' Now invert with respect to A, and we have: 'Through a fixed point A a line is drawn meeting a fixed conic in P, Q and S is a point on PQ such that $1/AP+1/AQ=2/AS$; find the locus of S.' It is a straight line (the polar of A). Hence the locus of the middle point of the chord of the bicircular quartic is a circle through A; and the locus of S for the original quartic is a conic through A, B, C.]

Ex. 2. The four points of contact of tangents from two nodes of a trinodal quartic lie on a conic through those nodes.

[Projecting the two nodes into the circular points and inverting with respect to the other node, we have: 'The intersections of a conic with a pair of directrices are concyclic.']

Ex. 3. A quartic has nodes A, B, C. Tangents BM, BM' and CN, CN' are drawn to the quartic. Show that two bitangents to the quartic pass through the intersection of MN and $M'N'$.

If BM and BM' meet CA in m and m', CN and CN' meet BA in n and n', then two bitangents pass through the intersection of mn and $m'n'$.

[Project B, C into the circular points and invert with respect to A; or use the equation of the bitangents given in § 3. These theorems are due to Jolliffe, *Messenger Math.*, xxxiii.]

Ex. 4. Six conics pass through the nodes A, B, C of a trinodal quartic and touch the curve. The first conic meets the second again in P, the second meets the third again in Q, and so on for the points P, Q, R, S, T, U. Show that the conics $ABCPS$, $ABCQT$, $ABCRU$ pass through the same four points.

[Projecting and inverting we have Brianchon's theorem. Derive a theorem from Pascal's theorem similarly.]

Ex. 5. A quartic has three nodes A, B, C. The conics through A, B, C osculating the curve at A cut at L and M a conic through A, B, C touching the curve at P. Show that LM, PA are conjugate chords of the conic $LMPABC$.

Ex. 6. A quartic has three nodes A, B, C. A chord PQ meets BC in R and (PQ, RS), $A(PQ, BC)$ are harmonic. Show that the locus of S is a conic through B, C and that the envelope of the conic $ABCPQ$ is a quartic with A as node and B, C as cusps.

Ex. 7. A quartic has nodes A, B, C. Conics are drawn through these nodes touching the quartic at P and Q. If PQ passes through A, the locus of the remaining intersection of the conics is a conic through A, B, C.

Ex. 8. Quartics with three given nodes and passing through four other fixed points cut any line through a node in involution.

Ex. 9. In general there are two quartics with three given nodes, passing through four other fixed points and touching the circle through the nodes (the point of contact not being a node).

Ex. 10. A quartic has a node A and two cusps B, C. Prove that

(i) If AD is the chord conjugate to BC of any conic through A, B, C touching the quartic, the locus of D is a conic through B and C.

(ii) If tangents from B and C to the curve meet at D and AD meets the curve at P, PA and the tangent at P divide BC harmonically.

(iii) The chord of contact of a conic through B and C touching the quartic at two points passes through one or other of two fixed points.

(iv) If PAQ is a chord of the quartic, the conics through A, B, C touching the curve at P and Q meet again on a fixed conic through A, B, C.

[Project B and C into the circular points and invert with respect to A. The reader may derive other properties of the quartic similarly, or may obtain other theorems by reciprocating those given above.]

Ex. 11. The tangents at the cusps of a tricuspidal quartic are concurrent.

[The reciprocal of: 'The inflexions of a nodal cubic are collinear.' Obtain other properties of the tricuspidal quartic by reciprocating the examples in Ch. XIII, § 4.

The result is also evident on projecting the quartic into a cardioid or three-cusped hypocycloid.]

Ex. 12. Quartics have given cusps A, B, C and the tangent at A is also given. Show that the locus of the point of contact of a tangent drawn from a fixed point on BC is a pair of lines through A.

Show also that the tangents to two such quartics at an intersection divide BC harmonically.

Ex. 13. Through the cusps of a tricuspidal quartic three conics are drawn touching the curve elsewhere. Show that the remaining intersections of the conics are collinear.

Ex. 14. Find the envelope of the line through any point P of a tricuspidal quartic which forms a harmonic pencil with the lines joining P to the cusps.

Ex. 15. Any tangent to a tricuspidal quartic meets the curve again at P and Q. The tangents at P, Q meet at T and cut the bitangent IJ at E, F. Prove that

(i) (EF, IJ) is harmonic.

(ii) The locus of T is a conic through I, J touching the quartic at three points.

(iii) The line joining T to the intersection of PF and QE passes through the point of concurrency of the cuspidal tangents.

Ex. 16. Two bicircular quartics with the same foci and common finite node cut orthogonally.

Ex. 17. If S and S' are the foci of a bicircular quartic with a finite node A, $(AS'.SP \pm AS.S'P)/AP$ is constant, P being any point on the curve.

Ex. 18. In Ex. 17 the lines AS, AS' are equally inclined to the tangents from A to the quartic.

Ex. 19. A bicircular quartic with finite node O can be regarded as the envelope of a circle passing through O whose centre lies on a fixed conic. It can also be regarded in two ways as the envelope of a circle orthogonal to a fixed circle j through O whose centre lies on a fixed conic touching j at O.

§ 3. The Quartic $\dfrac{a}{x^2}+\dfrac{b}{y^2}+\dfrac{c}{z^2}+\dfrac{2f}{yz}+\dfrac{2g}{zx}+\dfrac{2h}{xy}=0.$

Another method of obtaining properties of the trinodal quartic is to take the three double points as vertices of the triangle of reference. In the equation of the quartic there can be no terms involving the third or fourth power of x, y, or z. Hence the quartic takes the form

$$ay^2z^2+bz^2x^2+cx^2y^2+2xyz(fx+gy+hz)=0 \quad . \quad . \text{ (i).}$$

If we divide by $x^2y^2z^2$, we get the equation at the head of this section.

We shall denote by A, B, C, F, G, H the cofactors of a, b, c, f, g, h in

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

so that $A=bc-f^2$, $F=gh-af$, &c.

The tangents at the nodes are

$$cy^2+2fyz+bz^2=0,\quad az^2+2gzx+cx^2=0,\quad bx^2+2hxy+ay^2=0,$$

being the terms multiplying x^2, y^2, z^2 respectively equated to zero.

The tangents from the nodes touching the quartic elsewhere are

$$By^2-2Fyz+Cz^2=0,\quad Cz^2-2Gzx+Ax^2=0,$$
$$Ax^2-2Hxy+By^2=0.$$

For the quartic may be put in the form

$$\left(\frac{a}{x}+\frac{h}{y}+\frac{g}{z}\right)^2+\frac{C}{y^2}-\frac{2F}{yz}+\frac{B}{z^2}=0 \quad . \quad . \quad . \text{ (ii);}$$

which shows that the lines $By^2-2Fyz+Cz^2=0$ touch the quartic at its intersections with the conic $a/x+h/y+g/z=0$ other than the nodes.

The four bitangents of the quartic are

$$fx+gy+hz=\pm\sqrt{bc}\,x\pm\sqrt{ca}\,y\pm\sqrt{ab}\,z,$$

either three or one + signs being taken on the right.

For putting

$$u\equiv fx+gy+hz,$$

the equation of the quartic may be written

$$(\sqrt{a}\,yz+\sqrt{b}\,zx+\sqrt{c}\,xy)^2$$
$$=2xyz\,(-u+\sqrt{bc}\,x+\sqrt{ca}\,y+\sqrt{ab}\,z)\quad . \quad . \quad \text{(iii)};$$

showing that

$$u=\sqrt{bc}\,x+\sqrt{ca}\,y+\sqrt{ab}\,z$$

is a bitangent with its points of contact on

$$\sqrt{a}\,yz+\sqrt{b}\,zx+\sqrt{c}\,xy=0;$$

and similarly for the other bitangents.

The result may also be proved by writing the equation of the quartic in the form

$$(u^2-bc\,x^2-ca\,y^2-ab\,z^2)^2$$
$$=(u-\sqrt{bc}\,x-\sqrt{ca}\,y-\sqrt{ab}\,z)\,(u-\sqrt{bc}\,x+\sqrt{ca}\,y+\sqrt{ab}\,z)$$
$$\times(u+\sqrt{bc}\,x-\sqrt{ca}\,y+\sqrt{ab}\,z)\,(u+\sqrt{bc}\,x+\sqrt{ca}\,y-\sqrt{ab}\,z)$$
$$. \quad . \quad . \quad \text{(iv)}.$$

The tangential equation of the quartic, found in the usual way,* is

$$\Sigma^2(\Delta\Sigma+\Lambda^2)=\lambda\mu\nu\,(18\,\Delta\Lambda\Sigma+16\Lambda^3+27\,\Delta^2\lambda\mu\nu),$$

where $\qquad \Lambda\equiv F\lambda+G\mu+H\nu$

and $\qquad \Sigma\equiv a\lambda^2+b\mu^2+c\nu^2-2f\mu\nu-2g\nu\lambda-2h\lambda\mu.$

It will be found that

$$\Delta\Sigma+\Lambda^2\equiv BC\lambda^2+CA\,\mu^2+AB\nu^2+2AF\mu\nu+2BG\nu\lambda+2CH\lambda\mu$$
$$. \quad . \quad . \quad . \quad \text{(v)}.$$

The cross-ratio ϕ of the range in which the line

$$\lambda x+\mu y+\nu z=0$$

is divided by the quartic is given by

$$I^3\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2=27J^2(\phi^2-\phi+1)^3$$

where

$$I\equiv 3\,\Sigma^2-36\lambda\mu\nu\Lambda,\quad J\equiv-\Sigma^3+18\,\lambda\mu\nu\Sigma\Lambda+54\,\Delta\lambda^2\mu^2\nu^2.$$

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* See Ch. IV, § 3. Or, if $\lambda x+\mu y+\nu z=0$ and the quartic touch one another, so do $\lambda/x+\mu/y+\nu/z=0$ and $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$. Now write down the well-known condition that these two conics should touch.

For the lines joining the intersections of the line and quartic to (0, 0, 1) are

$$6b\lambda^2x^4+12(h\lambda^2+b\lambda\mu-f\lambda\nu)x^3y$$
$$+6(a\lambda^2+b\mu^2+c\nu^2-2f\mu\nu-2g\nu\lambda+4h\lambda\mu)x^2y^2$$
$$+12(h\mu^2+a\lambda\mu-g\mu\nu)xy^3+6a\mu^2y^4=0;$$

and the cross-ratio ϕ of this pencil is given by the above equation (Ch. I, § 11).

Ex. 1. If a quartic with three real nodes has a real bitangent, each node is either an acnode, or is a crunode such that the tangents at this crunode and the lines joining it to the other two nodes form two non-overlapping pairs of lines. In this case all four bitangents are real, the tangents from the nodes are all real or all unreal, and the quartic may be put in the form

$$y^2z^2+z^2x^2+x^2y^2+2xyz(fx+gy+hz)=0.$$

[For a real bitangent a, b, c have the same sign.]

Ex. 2. A real quartic with three nodes of which only one is real has two real bitangents.

[We may suppose a and b, f and g, x and y conjugate imaginaries, and c, h, z real.]

Ex. 3. The envelope of a line divided by a trinodal quartic in a equianharmonic range is a curve of the fourth class with four nodes. Three of these nodes are the nodes of the quartic and the two curves have the same tangents at these nodes. The other common tangents of the curves are the inflexional tangents of the quartic.

[The envelope is $I=0$. The fourth node is (F, G, H). We shall show in § 4 that $\Sigma=0$ touches the nodal tangents of the quartic.]

Ex. 4. The envelope of a line divided harmonically by a trinodal quartic is of the sixth class. It has the nodes of the quartic as biflecnodes, the two curves having the same tangents at these nodes. The other common tangents of the curves are the inflexional tangents of the quartic.

[The envelope is $J=0$.]

Ex. 5. Show that $(\Sigma I+3J)/\lambda\mu\nu=0$ is a curve of the third class and that the twelve common tangents of this curve and $I=0$ are the nodal and inflexional tangents of the quartic.

Ex. 6. If one of the nodes becomes a cusp. one of A, B, C is zero.

Ex. 7. What modification occurs in the tangential equation of the curve, if one or more nodes become cusps?

Ex. 8. Find the bitangents if one or more of the nodes becomes a cusp. What modification must be made in Ex. 1 in this case?

Ex. 9. The diagonals of the quadrilateral formed by the bitangents are $Gx+Fy-Cz=0$, &c. The triangle whose sides are these diagonals and the triangle whose vertices are the nodes are in plane perspective.

[The axis of perspective is $x/F+y/G+z/H=0$]

Ex. 10. The four conics through A, B, C and the points of contact of a bitangent touch in pairs at C, and the two tangents at C are harmonic conjugates for CA and CB.

[The conics are $\pm\sqrt{a}\,yz \pm \sqrt{b}\,zx \pm \sqrt{c}\,xy = 0$.]

Ex. 11. The equation $\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s} = 0$, where p, q, r, s are of the first degree in x, y, z, represents a quartic with bitangents

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0,$$

and with nodes

$$p = s, \ q = r; \quad q = s, \ r = p; \quad r = s, \ p = q.$$

Conversely, the equation of any trinodal quartic can be put in this form.

[The equation may be put in either of the forms

$$\{2(p^2+q^2+r^2+s^2)-(p+q+r+s)^2\}^2 = 64pqrs,$$

$$\{(p-s)+(q-r)\}^4-8(pq+rs)\,\{(p-s)+(q-r)\}^2 + 16\,\{q(p-s)+s(q-r)\}^2 = 0.$$

For the converse take $p \equiv u-\sqrt{bc}\,x+\sqrt{ca}\,y+\sqrt{ab}\,z$, &c., in § 3 (iv).

The points of contact of the four bitangents all lie on the conic

$$2(p^2+q^2+r^2+s^2) = (p+q+r+s)^2.]$$

Ex. 12. Three bitangents of a quartic with nodes A, B, C form a triangle PQR. Show that the triangles ABC and PQR are in perspective.

Show that, if AP meets QR in P', &c., the conic touching QR at P', RP at Q', PQ at R' passes through the points of contact of the fourth bitangent.

[The centre of perspective is $p = q = r$, and the conic is

$$2(p^2+q^2+r^2) = (p+q+r)^2.]$$

Ex. 13. The four centres of perspective obtained in Ex. 12 are the vertices of a quadrangle whose harmonic triangle is ABC.

[For the quartic of § 3 (i) they are $(\pm\sqrt{a},\ \sqrt{\pm b},\ \pm\sqrt{c})$.]

Ex. 14. One of the common chords of any two of the four conics of Ex. 12 passes through a node.

Ex. 15. Find the locus of the nodes of a trinodal quartic, given the four bitangents and two points on the curve.

Ex. 16. The double rays of the involution formed by the tangents at C and the lines CA, CB pass through a pair of vertices of the quadrilateral formed by the bitangents.

[$bx^2 = ay^2$.]

Ex. 17. The bitangents touch the quartic at real points if a, b, c are positive and

$$2(af^2+bg^2+ch^2-abc) > (\pm\sqrt{a}\,f \pm \sqrt{b}\,g \pm \sqrt{c}\,h - \sqrt{abc})^2,$$

three or one + signs being taken.

Ex. 18. The quartic is the envelope of the conic

$$Al^2x^2+Bm^2y^2+Cn^2z^2+2Fmnyz+2Gnlzx+2Hlmxy = 0,$$

where l, m, n vary subject to $l+m+n = 0$.

Ex. 19. The quartic is the envelope of the conic

$$t^2x^2+2t(ayz+hzx+gxy)-By^2+2Fyz-Cz^2 = 0,$$

where t varies; and of two similar families of conics.

Ex. 20. *ABC* is a fixed triangle and *O* a fixed point. Through the remaining intersections of *AO*, *BO*, *CO* with the sides of the triangle a conic of given eccentricity is drawn meeting the sides of the triangle again at *D*, *E*, *F*. Show that *AD*, *BE*, *CF* meet in a point whose locus is a quartic with nodes at *A*, *B*, *C*.

Ex. 21. Conics of given eccentricity pass through fixed points *A*, *B*, *C*. Show that their envelope and the locus of their centres are quartics with nodes at *A*, *B*, *C*.

[The envelope of an asymptote or axis is a three-cusped hypocycloid. See *Annals of Math.*, II. iii (1902), p. 154; *Trans. Amer. Math. Soc.*, iv (1903), pp. 103, 489.]

Ex. 22. A conic touches the sides of a given triangle *ABC*, and one focus lies on a fixed conic *S*. Show that the locus of the other focus is a quartic with nodes at *A*, *B*, *C*.

Ex. 23. Given the nodes and five other points of a unicursal quartic, construct the tangents at and from the nodes and any number of other points on the curve.

[Use Ex. 22. By varying the relative positions of *S* and *ABC*, we may find every possible shape of a quartic with three real double points.]

§ 4. Conics connected with a Trinodal Quartic.

There are many conics of interest connected with the trinodal quartic

$$ay^2z^2+bz^2x^2+cx^2y^2+2xyz\,(fx+gy+hz)=0 \quad . \quad . \quad \text{(i)}.$$

Their properties are given in the following theorems.

If the point-equation of the conic is obtained, the tangential equation can, of course, be at once deduced, and vice versa. Various forms of the equation of each conic are given, as they are needed in the examples.

To save space we use the following contractions:

$$\left\{\begin{aligned}
u &\equiv fx+gy+hz,\\
\Lambda &\equiv F\lambda+G\mu+H\nu,\\
M &\equiv f(gG+hH)x+g(hH+fF)y+h(fF+gG)z,\\
\Sigma &\equiv a\lambda^2+b\mu^2+c\nu^2-2f\mu\nu-2g\nu\lambda-2h\lambda\mu,\\
U &\equiv afyz+bgzx+chxy,\\
\Upsilon &\equiv a^2f^2\lambda^2+b^2g^2\mu^2+a^2h^2\nu^2-2bcgh\,\mu\nu-2cahf\nu\lambda\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -2\,abfg\lambda u,\\
T &\equiv ghyz+hfzx+fgxy,\\
K &\equiv ay^2z^2+bz^2x^2+cx^2y^2+2xyz\,(fx+gy+hz).
\end{aligned}\right.$$

By u, Λ, Σ we denote the same as in § 3. The identity of §3 (v) is useful in the examples.

The conics $T=0$ and $U=0$ (i.e. $\Upsilon=0$) are of importance n the theory. They pass through the nodes of the quartic

$K = 0$ and meet the quartic again at its intersections with the line $M = 0$. For we have

$$TU + xyzM \equiv fghK.$$

I. *The eight points of contact of the four bitangents lie on the conic*

$$u^2 = bcx^2 + cay^2 + abz^2$$

i. e.

$$(abc - af^2 - bg^2 - ch^2)(a\lambda^2 + b\mu^2 + c\nu^2) + (af\lambda + bg\mu + ch\nu)^2 = 0.$$

This follows at once from § 3 (iv). See also § 3, Ex. 11.

II. *The six intersections of the tangents at a node with the line joining the other two nodes lie on the conic*

$$bcx^2 + cay^2 + abz^2 + 2afyz + 2bgzx + 2chxy = 0,$$

or $$u^2 = Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy;$$

i. e.

$$a^2A\lambda^2 + b^2B\mu^2 + c^2C\nu^2 + 2bcF\mu\nu + 2caG\nu\lambda + 2abH\lambda\mu = 0,$$

or $$abc\Sigma = \Upsilon.$$

For the intersections are $z = 0$, $bx^2 + 2hxy + ay^2 = 0$, &c.

III. *The six tangents at the nodes touch the conic*

$$(Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy) + 4T = 0,$$

or $$(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy) + 4U = 0;$$

i. e. $$\Sigma = 0.$$

This result is at once established by using the equations of the nodal tangents; or it may be deduced from II by using the fact that the lines joining the vertices of the triangle of reference to the intersections of the conic

$$\mathbf{a}x^2 + \mathbf{b}y^2 + \mathbf{c}z^2 + 2\mathbf{f}yz + 2\mathbf{g}zx + 2\mathbf{h}xy = 0$$

with the opposite sides all touch the conic

$$\mathbf{bc}\lambda^2 + \mathbf{ca}\mu^2 + \mathbf{ab}\nu^2 - 2\mathbf{af}\mu\nu - 2\mathbf{bg}\nu\lambda - 2\mathbf{ch}\lambda\mu = 0.$$

IV. *The six intersections of the tangents from a node with the line joining the other two nodes lie on the conic*

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = 0;$$

i. e. $$\Delta\Sigma + 2(GH\mu\nu + HF\nu\lambda + FG\lambda\mu) = 0.$$

For the intersections are $z = 0$, $Cx^2 - 2Hxy + By^2 = 0$, &c.

V. *The six tangents from the nodes touch the conic*

$$aA^2x^2 + bB^2y^2 + cC^2z^2 + 2fBCyz + 2gCAzx + 2hABxy = 0;$$

i. e. $$\Delta\Sigma + \Lambda^2 = 0.$$

VI. *The six inflexional tangents of the quartic touch*

$$4(aA^2x^2+\ldots+\ldots+2fBCyz+\ldots+\ldots)$$
$$=\Delta(Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy)+4\Delta U;$$

i.e. $$3\Delta\Sigma+4\Lambda^2=0.$$

For, if $$\lambda x+\mu y+\nu z=0$$
is an inflexional tangent, we obtain on eliminating z between the equations of this line and the quartic an equation in x/y with three equal roots. The same will therefore be true, if we eliminate z between the result of substituting $1/x$ for x, $1/y$ for y, $1/z$ for z in the equations of the line and quartic.

This shows that the conics
$$\lambda/x+\mu/y+\nu/z=0,\quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$
have three-point contact.

The conditions for this are well known to be (with the notation of Salmon's *Conic Sections*)
$$3\Delta/\Theta=\Theta/\Theta'=\Theta'/3\Delta',$$
where $$\Delta'=2\lambda\mu\nu,\quad \Theta'=-\Sigma,\quad \Theta=2\Lambda.$$

The relation $\Theta^2=3\Delta\Theta'$ is the tangential equation of the conic required.

VII. *The six inflexions of the quartic lie on the conic*

$$2(aA^2x^2+\ldots+\ldots+2fBCyz+\ldots+\ldots)$$
$$-2\Delta(Ax^2+By^2+Cz^2-Fyz-Gzx-Hxy)+\Delta U=0,$$

or

$$2M(ghAx+hfBy+fgCz)$$
$$=U\{fgh(4abc+2fgh-\Delta)-2bcg^2h^2-2cah^2f^2-2abf^2g^2\}.$$

Since with the notation of the last paragraph the two conics (ii) have three-point contact, therefore for some value of k

$$k(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)$$
$$-\{x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')\}$$
$$(px+qy+rz)$$
$$\equiv\lambda yz+\mu zx+\nu xy;$$

where (x', y', z') is the point of contact of the conics and $px+qy+rz=0$ is the line joining this point of contact to their fourth point of intersection. Equating to zero the coefficients of x^2, y^2, z^2 on the left-hand side, and remembering that $px'+qy'+rz'=0$, we have

$$\frac{ax'}{ax'+hy'+gz'}+\frac{by'}{hx'+by'+fz'}+\frac{cz'}{gx'+fy'+cz'}=0.$$

Hence the point of osculation of the two conics lies on

$$ax(hx+by+fz)(gx+fy+cz)+\ldots+\ldots=0,$$

and therefore on

$$\begin{aligned}(ghx+hfy+fgz)&(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)\\ &=ax(hx+by+fz)(gx+fy+cz)+\ldots+\ldots,\end{aligned}$$

i.e. on

$$\begin{aligned}2Ayz(hy+gz)+2Bzx(fz+hx)&+2Cxy(gx+fy)\\ &+(4abc-4fgh-\Delta)xyz=0.\end{aligned}$$

Replacing now x by $1/x$, y by $1/y$, z by $1/z$, we see that the point of contact of $\lambda x+\mu y+\nu z=0$ and the quartic, i.e. any inflexion of the quartic, lies on

$$\begin{aligned}2Ax^2(gy+hz)+2By^2(hz+fx)&+2Cz^2(fx+gy)\\ &+(4abc-4fgh-\Delta)xyz=0.\end{aligned}$$

Of the twelve intersections of this cubic with the quartic six lie on the conic $T=0$ which touches the cubic at each vertex of the triangle of reference. Therefore the remaining six intersections of the cubic and quartic, namely, the inflexions of the quartic, lie on a conic (Ch. XII, § 7). We can verify that

$$\begin{aligned}&(afghA+bfghB+cfghC+2bcghF+2cahfG+2abfgH)K\\ &\quad+M\{2Ax^2(gy+hz)+2By^2(hz+fx)+2Cz^2(fx+gy)\\ &\qquad\qquad+(4abc-4fgh-\Delta)xyz\}\\ &\equiv T\{2A(gG+hH)x^2+\ldots+\ldots\\ &\qquad\qquad-(4fF^2+2gh\Delta+3af\Delta)yz-\ldots-\ldots\}.\end{aligned}$$

The contents of the last brackets { } equated to zero give the equation of the conic through the inflexions, which is readily seen to be the same as that given above. This proof is due to Richmond and Stuart, *Proc. London Math. Soc.*, II. i (1903), p. 130.

The remaining intersections of the quartic with the inflexions-conic are those intersections of the quartic with the line $M=0$ which do not lie on the conic $T=0$; i.e. the intersections of the line with the conic $U=0$.

VIII. *The six points where the tangents at the nodes meet the quartic again lie on the conic*

$$\begin{aligned}2M(&bcghx+cahfy+abfgz)\\ &=U(5abcfgh-4fgh\Delta-2bcg^2h^2-2cah^2f^2-2abf^2g^2).\end{aligned}$$

The lines joining the intersections of the quartic with $\lambda x+\mu y+\nu z=0$ to the node (0, 0, 1) are

$$\begin{aligned}(ay^2+bx^2+2hxy)(\lambda x+\mu y)^2-2xy(fx+gy)\nu(\lambda x+\mu y)&\\ +c\nu^2x^2y^2=0.&\end{aligned}$$

If $ay^2+bx^2+2hxy=0$ is one pair of these lines and $Px^2+2Qxy+Ry^2=0$ is the other pair, we have readily, on identifying the four lines with

$$(ay^2+bx^2+2hxy)\,(Px^2+2(Qxy+Ry^2)=0,$$

$$\lambda:\mu:\nu\ =gbc:fca:-2\,(fF+gG);$$

$$P:Q:R=b^2cg^2:fg\,(abc+4fgh-2af^2-2bg^2):a^2cf^2.$$

Suppose the tangents at the nodes meet the curve again in A_1 and A_2, B_1 and B_2, C_1 and C_2. We have just shown that C_1C_2 is the line

$$\frac{bc}{f}x+\frac{ca}{g}y=\frac{2\,(fF+gG)}{fg}z.$$

Now, since

$$(bz^2+cy^2+2fyz)\,(az^2+cx^2+2gzx)-cK$$
$$\equiv z^2\{abz^2+2\,(2fg-ch)\,xy+2bgzx+2afyz\},$$

the four points A_1, A_2, B_1, B_2 lie on the conic

$$abz^2+2\,(2fg-ch)\,xy+2bgzx+2afyz=0.$$

They therefore lie on the conic

$$k\,\{abz^2+2\,(2fg-ch)\,xy+2bgzx+2afyz\}$$
$$=\left\{-\frac{2\,(gG+hH)}{gh}x+\frac{ca}{g}y+\frac{ab}{h}z\right\}\left\{\frac{bc}{f}x-\frac{2\,(hH+fF)}{hf}y+\frac{ab}{h}z\right\}.$$

We require to show that k can be chosen so that this conic also passes through C_1 and C_2, in other words that k can be chosen so that the equation of this conic is symmetrical.

A comparison of the coefficients of x^2, y^2, z^2 shows that the required value of k must be

$$\{2h\,(fF+gG)+abfg\}\div fgh^2,$$

and straightforward verification then shows that for this value of k the conic reduces to

$$2bc\,(gG+hH)\,x^2+\ldots+\ldots$$
$$=a\,(abcf+2bcgh+4af^3-8f^2gh)\,yz+\ldots+\ldots,$$

which is equivalent to the given form.

IX. *The six points of contact of the tangents from the nodes to the quartic lie on the conic* *

$$aA^2x^2+bB^2y^2+cC^2z^2+2fBCyz+2gCAzx+2hABxy$$
$$=\Delta\,(Ax^2+By^2+Cz^2-Fyz-Gzx-Hxy).$$

---

* An alternative form is obtained by omitting the Δ from the right-hand side of the second form of the equation of the conic through the six inflexions.

If the tangents touch the quartic in A_1 and A_2, B_1 and B_2, C_1 and C_2, we prove as in the case of Theorem VIII that C_1C_2 is

$$\frac{A}{f}x+\frac{B}{g}y=\frac{fF+gG}{fg}z;$$

the lines joining $(0, 0, 1)$ to the intersections of this line and the quartic being the tangents from the nodes $Ax^2+By^2=2Hxy$ and the lines

$$bg^2Ax^2+2fg(\Delta-hH)xy+af^2By^2=0.$$

Then since

$$(By^2-2Fyz+Cz^2)(Ax^2-2Gzx+Cz^2)-\Delta K$$
$$\equiv(Cz^2-Fyz-Gxz+Hxy)^2,$$

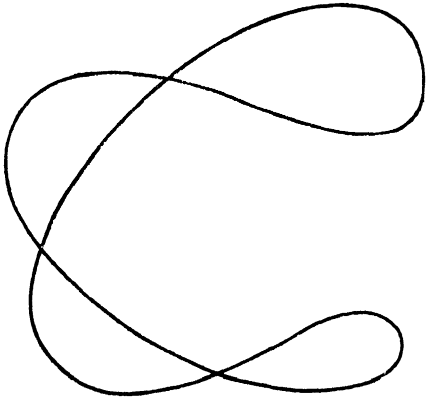

Fig. 3.
$3(y^2-x^2)(x-4)(4x-y-40)-10(x^2+2y^2-12x)^2.$

the points A_1, A_2, B_1, B_2 lie on the conic

$$Cz^2-Fyz-Gxz+Hxy=0$$

and also on

$$k(Cz^2-Fyz-Gxy+Hxy)$$
$$=\left(\frac{gG+hH}{gh}x-\frac{B}{g}y-\frac{C}{h}z\right)\left(-\frac{A}{f}x+\frac{hH+fF}{hf}y-\frac{C}{h}z\right),$$

which reduces to the given form on taking

$$k=(fhF+ghG+fgC)/fgh^2.$$

In the following examples the conic through the points of contact of the bitangents, which is referred to in Theorem I, is called 'Conic I'; and so for the other theorems.

In Figs. 3, 4 are shown typical unicursal quartics.

Ex. 1. The tangents *at* the nodes A, B, C of a trinodal quartic meet the curve again in A_1 and A_2, B_1 and B_2, C_1 and C_2. Show that the intersections of A_1A_2 and BC, B_1B_2 and CA, C_1C_2 and AB are collinear. Prove also a similar result for the points of contact of the tangents *from* the nodes.

[The required lines are $bcghx + cahfy + abfgz = 0$ and
$$ghAx + hfBy + fgCz = 0.]$$

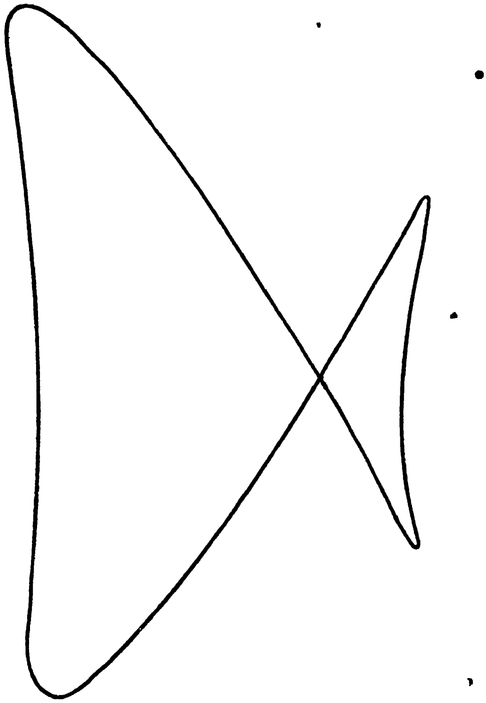

Fig. 4.
$(y^2 - 4x^2)(x - 4)(20x + y + 160) - 10(2x^2 - y^2 + 8x)^2$.

Ex. 2. Show that if in Ex. 1 (either case) the line A_1A_2 meets the quartic again in A'_1, A'_2, &c., the six lines AA'_1, AA'_2, BB'_1, BB'_2, CC'_1, CC'_2 touch a conic, and their intersections with BC, CA, AB respectively lie on a conic.

[The equations of the lines are given in § 4.]

Ex. 3. The conics III, V, VI touch at the same two points.

[Evident from their tangential equations. The chord of contact is
$$AFx + BGy + CHz = 0,$$
and its pole is (F, G, H).]

Ex. 4. The conics I, VI, VII pass through the same four points.

[If $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are the point-equations of the conics in the first form given, we have $S_3 - 2S_2 \equiv 3\Delta S_1$. This result is due to W. Gross; see Stahl, *Crelle*, civ, p. 308.]

Ex. 5. The conics $U = 0$, VII, IX pass through the same four points.

Ex. 6. The conics $U = 0$, I, IV pass through the same four points.

Ex. 7. The conics $T = 0$, III, IV pass through the same four points.

Ex. 8. The conics $U = 0$, VII, VIII, IX pass through the same two points on the quartic.

[The intersections of $U = 0$ with $M = 0$.
Obtain theorems by projecting these points into the circular points, inverting with respect to a node, and generalizing by projection.]

Ex. 9. The conics $U = 0$, II, III have four common tangents.

Ex. 10. The conics II and IV have double contact. The chord of contact passes through the intersection of a common chord of $U = 0$ and VII, and a common chord of $U = 0$ and VIII.

Ex. 11. A conic can be drawn touching the lines joining the nodes and the common tangents of the conics III and IV.

Ex. 12. A conic can be drawn having the nodes as vertices of a self-conjugate triangle and touching the common tangents of the conics IV and V.

Ex. 13. The diagonals of the quadrilateral formed by the bitangents are the polars of the nodes with respect to the conic IV.

Ex. 14. If the conic

$$Ax^2 + By^2 + Cz^2 - Fyz - Gzx - Hxy = 0$$

meets the sides of the triangle of reference in X_1 and X_2, Y_1 and Y_2, Z_1 and Z_2, the lines AX_1 and AX_2, &c., meet the conic again at its intersections with the diagonals of the quadrilateral formed by the bitangents. The conic passes through the intersections of the conics V and IX.

Ex. 15. Show that the common tangents of the quartic and the conic $\Delta\Sigma + k\Lambda^2 = 0$ are the common tangents of the conic and the two curves of the third class

$$(1-k)\,\Lambda^3 = \{(8-9k) \pm (4-3k)^{\frac{3}{2}}\}\,\lambda\mu\nu.$$

Deduce the equations of the conics III, V, VI.

[$k = 0, 1, 4/3$. Use the tangential equation of the quartic.]

Ex. 16. Conics are drawn touching the four bitangents and one side of the triangle ABC. Show that the points of contact with these sides are collinear.

[On $AFx + BGy + CHz = 0$. Any conic touching the bitangents is

$$\{(bg^2 - ch^2)\,\lambda^2 - bA\mu^2 + cA\nu^2 + 2cG\nu\lambda - 2bH\lambda\mu\}$$
$$+ k\,\{-aB\lambda^2 + (af^2 - ch^2)\,\mu^2 + cB\nu^2 + 2cF\nu\mu - 2aH\lambda\mu\} = 0.]$$

Ex. 17. If the points of contact of the tangents from P lie on a conic, the locus of P is the cubic

$$Fx\,(cy^2 - bz^2) + Gy\,(az^2 - cx^2) + Hz\,(bx^2 - ay^2) = 0.$$

[The polar cubic of P touches a conic at A, B, C.]

Ex. 18. If the tangents at A are harmonic conjugates with respect to AB and AC, the points of contact of tangents from C lie on a line through B, and so do the remaining intersections of the curve with the tangents at C.

[$f = 0$.]

Ex. 19. If the tangents from A are harmonic conjugates with respect to AB and AC, the points of contact of tangents from B and C are collinear.

[$F = 0$.]

Ex. 20. The tangents at the nodes meet by threes in two points, if $\Delta = 4fgh$.

[Conic III is a point-pair. Illustrate by tracing

$$x^2y^2 - 2xy(x-2y) - 3x^2 + 8xy + 3y^2 = 0.]$$

Ex. 21. The tangents from the nodes cannot meet by threes in two points.

Ex. 22. The points of contact of the four bitangents are the intersections of the quartic with two lines, if $\Delta = 2fgh$.

[Conic I is a line pair.]

Ex. 23. The intersections of the tangents at each node with the line joining the other two nodes cannot lie on two lines.

Ex. 24. The intersections of the tangents from each node with the line joining the other two nodes lie on two lines, if $\Delta^2 = 4FGH$.

Ex. 25. Show that conics II and V are always real. What conditions must hold in order that conics I, III, IV may be real?

[The conic $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is real, if $a\Delta$ and C are not both positive.]

Ex. 26. Conic II for

$$ay^2z^2 + \ldots + \ldots + 2fx^2yz + \ldots + \ldots = 0$$

is the polar reciprocal with respect to $x^2 + y^2 + z^2 = 0$ of conic V for

$$Ay^2z^2 + \ldots + \ldots + 2Fx^2yz + \ldots + \ldots = 0,$$

and vice versa.

Similarly for conics III and IV.

The reader should read Ch. XVIII, §§ 1 and 4, before attempting the following examples.

Ex. 27. The tangents at A and B form a quadrilateral. One diagonal is AB; the other two meet at C' and intersect AB in R_1, R_2. Similarly we obtain points A', P_1, P_2 and B', Q_1, Q_2. Show that AA', BB', CC' are concurrent. Show also that three of the points P_1, P_2, Q_1, Q_2, R_1, R_2 (say P_2, Q_2, R_2) are collinear, and that AP_1, BQ_1, CR_1 are concurrent.

[AC and AB are harmonically conjugate with respect to the tangents at A. Therefore AC' is $cy + fz = 0$, and similarly BC' is $cx + gz = 0$. Hence C' is $(g, f, -c)$, and the point of concurrency is $fx = gy = hz$. R_1, R_2 are $Ax^2 = By^2$, $z = 0$.]

Ex. 28. Show that AC' and the harmonic conjugate of BC' for BA, BC meet on a diagonal of the quadrilateral formed by the bitangents.

[See § 3, Ex. 9.]

Ex. 29. The lines BA', CA', CB', AB', AC', BC' touch a conic.

$[a\lambda^2 + \ldots + \ldots - (f + bc/f)\,\mu\nu + \ldots + \ldots = 0.]$

Ex. 30. The conic through ABC touching AC' and BC' passes through the points of contact of the tangents from C to the quartic.

$[gyz + fzx + cxy = 0.]$

Ex. 31. The points of contact of the tangents from C' to the quartic lie on a conic through C.

$[(fh + bg)\,xz + (gh + af)\,yz = Ax^2 + By^2 + 2\,(ch - 2fg)\,xy.]$

Ex. 32. The points of contact of the tangents from A lie on a conic through A, C and the intersections of BC' with the quartic. Similarly for the tangents from B.

$[bgxz + ghyz = By^2 + (ch - 2fg)\,xy\,;\ fhxz + afyz = Ax^2 + (ch - 2fg)\,xy.]$

Ex. 33. The conics of Ex. 31, 32 pass through the same four points.

Ex. 34. The tangents at C meet the quartic again in C_1 and C_2, and C_1C_2 meets the curve again at C'_1 and C'_2. Show that the points of contact of the tangents from C lie on a conic touching CC'_1 and CC'_2 at C'_1 and C'_2 and passing through A and B.

Show also that through the intersections of the conic $U = 0$ with this conic a conic can be drawn touching CA and CB at A and B.

$[(fF + gG + hH)\,z^2 - H\,(hz^2 + cxy) = cU.]$

Ex. 35. Through two pairs of vertices of the quadrilateral formed by the bitangents can be drawn (i) a conic touching $C'A$ at A and $C'B$ at B; (ii) a conic touching conic I at its intersections with AB; (iii) two conics each touching $C'A$ at A and $C'B$ at B and each passing through the points of contact of two bitangents.

$[$(i) $uz + cxy = 0$; (ii) $u^2 - bcx^2 - acy^2 + abz^2 = 0$;

(iii) $uz + cxy \pm \sqrt{ab}\,z^2 = 0.]$

Ex. 36. Show that, if

$$u \equiv fx + gy + hz,$$
$$v \equiv cxy + uz - tz^2,$$
$$w \equiv bcx^2 + acy^2 - u^2 + 2t\,(cxy + uz) - t^2z^2,$$

$w = 0$ touches the quartic at its four intersections with $v = 0$, other than A and B.

Show that $v = 0$ touches $C'A$ and $C'B$.

[The quartic is $v^2 + z^2w = 0.$]

Ex. 37. Show that $w = 0$ is a line-pair, if $t = h$ or if $t = \pm\sqrt{ab}$. Show that in the former case $w = 0$ is the tangents from C to the quartic, and that in the latter case $w = 0$ is two bitangents.

Ex. 38. Show that the locus of the pole of AB with respect to $w = 0$ is a conic through C, C', R_1, R_2; and that this and the two similar conics pass through the same four points.

$[Ax^2 - By^2 + Fyz - Gxz = 0.]$

Ex. 39. Show that $w = 0$ divides R_1R_2 harmonically.

§ 5. Tricuspidal Quartic.

The results of the preceding section require modification if the three double points of the quartic are not all ordinary nodes.

Suppose the double points are all cusps. Since the tangents at each double point are coincident, the quartic takes the form

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}\pm\frac{2\sqrt{(bc)}}{yz}\pm\frac{2\sqrt{(ca)}}{zx}\pm\frac{2\sqrt{(ab)}}{xy}=0.$$

The ambiguous signs may all be taken as *minus* without loss of generality. For unless the quartic is a pair of coincident conics, three plus signs or one plus and two minus signs are impossible; while if (say) the first sign is minus and the other two plus, we can make all signs minus by replacing $\sqrt{a}$ by $-\sqrt{a}$.

Finally, replacing x, y, z by $\sqrt{a}\,x$, $\sqrt{b}\,y$, $\sqrt{c}\,z$ we obtain the equation of the tricuspidal quartic in its canonical form

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{2}{yz}-\frac{2}{zx}-\frac{2}{xy}=0.$$

The cuspidal tangents $y=z$, $z=x$, $x=y$ are concurrent at the point (1, 1, 1).

Since the equation of the quartic may be written

$$(yz+zx+xy)^2=4xyz\,(x+y+z),$$

$x+y+z=0$ is the bitangent, its points of contact being its intersections with $yz+zx+xy=0$.

The equation of the quartic may be also written

$$x^{-\frac{1}{2}}+y^{-\frac{1}{2}}+z^{-\frac{1}{2}}=0.$$

The tangential equation is

$$(\lambda+\mu+\nu)^3=27\lambda\mu\nu.$$

In the equations of the three-cusped hypocycloid given in § 2 (ii) put

$$2X=a-x-\sqrt{3}\,y,\quad 2Y=a-x+\sqrt{3}\,y,\quad 2Z=a+2x;$$

so that $X=0$, $Y=0$, $Z=0$ are the sides of the triangle whose vertices are the three cusps.

This gives

$$6X=a\{1+2\cos(\phi+\tfrac{2}{3}\pi)\}^2,\quad 6Y=a\{1+2\cos(\phi+\tfrac{4}{3}\pi)\}^2,$$
$$6Z=a\{1+2\cos\phi\}^2,$$

from which is readily deduced

$$X^{-\frac{1}{2}}+Y^{-\frac{1}{2}}+Z^{-\frac{1}{2}}=0.$$

It follows that, if any three-cusped quartic is projected so that the cusps and the point of concurrency of the cuspidal

tangents become the vertices and centroid of an equilateral triangle, the quartic becomes a three-cusped hypocycloid.

Ex. 1. The coordinates of any point on the tricuspidal quartic in canonical form are expressed rationally in terms of a parameter t by

$$(t-1)^2x = (t+1)^2y = 4z.$$

Ex. 2. A conic is inscribed in a given triangle and passes through a fixed point. The locus of the point of concurrency of the lines joining the vertices of the triangle to the points of contact of the opposite sides is a tricuspidal quartic.

[If the point is (X, Y, Z) and the conic is

$$(x/\xi)^{\frac{1}{2}}+(y/\eta)^{\frac{1}{2}}+(z/\zeta)^{\frac{1}{2}} = 0,$$

the quartic is

$$(X/x)^{\frac{1}{2}}+(Y/y)^{\frac{1}{2}}+(Z/z)^{\frac{1}{2}} = 0.]$$

Ex. 3. A variable cubic touches three fixed lines at fixed collinear points and has inflexions at its other intersections with the fixed lines. Show that the line of these inflexions envelops a tricuspidal quartic, having the line of the fixed points as bitangent.

[Writing down the conditions that $(0, -\nu, \mu)$ and $(-\nu, 0, \lambda)$ are inflexions of

$$xyz+(x+y+z)^2(\lambda x+\mu y+\nu z) = 0,$$

we obtain

$$\mu^2\nu+\mu\nu^2+\nu^2\lambda+\nu\lambda^2+\lambda^2\mu+\lambda\mu^2 = 6\lambda\mu\nu\ ;$$

which is of the third class and fourth degree since it has (1, 1, 1) as a bitangent.]

Ex. 4. The locus of the centre of a conic having a given equilateral triangle as self-conjugate triangle and having its asymptotes inclined at an angle $\tan^{-1}\frac{1}{2}\sqrt{3}$ is a three-cusped hypocycloid.

Ex. 5. A fixed radius of a circle and the tangent at its extremity intercept on a moving line a segment which is bisected by the circle. Show that the envelope of the moving line is a three-cusped hypocycloid.

Ex. 6. The envelope of the Simson (pedal) line of a given triangle is a three-cusped hypocycloid.

The tangents from a vertex of the triangle are the two sides and altitude of the triangle through that vertex.

[Writing down the condition in trilinear coordinates that the lines perpendicular to the sides of the triangle at their intersections with

$$\lambda\alpha+\mu\beta+\nu\gamma = 0$$

are concurrent, we get the tangential equation of the envelope

$$2(1+\cos A\ .\ \cos B\ .\ \cos C)\lambda\mu\nu = \Sigma\sin^2 A\ .\ \mu\nu(\mu\cos C+\nu\cos B)$$

which touches $\sin A\ .\ \alpha+\sin B\ .\ \beta+\sin C\ .\ \gamma = 0$ at the circular points. See Converse, *Annals of Math.*, II. iii (1904), p. 105, for an extension of this result.]

Ex. 7. The equation of a quartic with one real and two unreal cusps may be put in the form

$$(x^2+y^2-2xz)^2 = 4z^2(x^2+y^2).$$

The quartic can be projected into a cardioid by a real projection.

[Replace x, y in § 5 by $x \pm iy$. For other examples on the tricuspidal quartic see § 2.]

§ 6. Other Quartics with Three Distinct Double Points.

The point (0, 0, 1) is a cusp of

$$ay^2z^2+bz^2x^2+cx^2y^2+2xyz\,(fx+gy+hz)=0 \quad . \quad . \quad \text{(i)}$$

if $C=0$, i.e. $ab=h^2$.

It is a flecnode if the terms

$$ax^2+2hxy+by^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

multiplying z^2 in (i) have a factor in common with the terms

$$xy\,(fx+gy) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

multiplying z. This is the case, if

$$fF+gG=0, \text{ i.e. } 2fgh=af^2+bg^2.$$

It is a biflecnode if (ii) is a factor of (iii), which is only possible if

$$f=g=0.$$

I. Quartics with three Biflecnodes.

Ex. 1. If two of the nodes of a trinodal quartic are biflecnodes, so is the third.

$[f=g=h=0.]$

Ex. 2. If a quartic has biflecnodes A, B, C, the tangents at A are harmonic conjugates for AB, AC.

The tangents at A, B, C touch a conic for which ABC is a self-conjugate triangle.

Ex. 3. If a quartic has three real biflecnodes, one is an acnode, and the others are crunodes. Its equation can be put in the form

$$x^{-2}+y^{-2}=z^{-2}.$$

Ex. 4. If a quartic has three real biflecnodes, the acnode is the intersection of two diagonals of the quadrilateral formed by the tangents at the crunodes.

Ex. 5. Show that the 'cross-curve'

$$x^2y^2=a^2y^2+b^2x^2,$$

the 'carbon-point-curve'

$$x^2y^2=a^2y^2-b^2x^2,$$

and the 'hour-glass-curve'

$$a^2\,(y^2-b^2)^2=4b^2x^2\,(y^2+b^2)$$

are curves of this type.

Trace the curves, and show that the two former are the loci of intersection of lines parallel to the axes of an ellipse or hyperbola through the intersections of any tangent with the axes.

Ex. 6. The points of contact of the tangents from O to a quartic with three biflecnodes A, B, C lie on a conic which meets the quartic again on the polar line of O with respect to the quartic.

[Take the quartic as $x^{-2}+y^{-2}+z^{-2}=0$ and O as (ξ, η, ζ). Then use the identity

$$\begin{aligned}&\{\xi\,(\eta^2+\zeta^2)\,x+\,\dots\,+\,\dots\}\,\{\xi x\,(y^2+z^2)+\,\dots\,+\,\dots\}\\ &\qquad\qquad -(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2)\,(y^2z^2+z^2x^2+x^2y^2)\\ &\equiv\{\eta\zeta yz+\zeta\xi zx+\xi\eta xy\}\,\{(\eta^2+\zeta^2)\,x^2+\,\dots\,+\,\dots\,+\eta\zeta yz+\,\dots\,+\,\dots\}.]\end{aligned}$$

Ex. 7. If in Ex. 6 O lies on the quartic, the points of contact lie on a line whose envelope is a conic having ABC as a self-conjugate triangle.

[The conic of Ex. 6 degenerates into $x/\xi+y/\eta+z/\zeta=0$ and the polar line of O.]

Ex. 8. Any point of the curve $x^{-2}+y^{-2}=z^{-2}$ is $(\sec\phi, \operatorname{cosec}\phi, 1)$.

Four points with parameters $\phi_1, \phi_2, \phi_3, \phi_4$ lie on a conic through the crunodes, if $\phi_1+\phi_2+\phi_3+\phi_4=(2n-1)\pi$.

The four points are collinear, if in addition $t_1t_2t_3t_4=-1$, where $t_1=\tan\frac{1}{2}\phi_1$, &c.

Three points with parameters ϕ_1, ϕ_2, ϕ_3 are collinear if

$$t_1t_2t_3(t_2t_3+t_3t_1+t_1t_2)=t_1+t_2+t_3.$$

Ex. 9. The conics through the crunodes A, B of $x^{-2}+y^{-2}=z^{-2}$ osculating the quartic at P, Q, R, S meet the curve again at P', Q', R', S'. If A, B, P, Q, R, S lie on a conic, so do A, B, P', Q', R', S'.

Ex. 10. Two conics through the crunodes A, B of $x^{-2}+y^{-2}=z^{-2}$ meet the quartic again in P_1, Q_1, R_1, S_1 and P_2, Q_2, R_2, S_2. If A, B, P_1, Q_1, P_2, Q_2 lie on a conic, so do A, B, R_1, S_1, R_2, S_2.

Ex. 11. The sextactic points of $x^{-2}+y^{-2}=z^{-2}$ are

$$(\pm i, \pm 2^{-\frac{1}{2}}, 1), \quad (\pm 2^{-\frac{1}{2}}, \pm i, 1), \quad (\pm 1, \pm 1, 2^{\frac{1}{2}}).$$

[The method of Ch. X, § 2, Ex. 17, gives $\sin 4\phi=0$ or $\cos 4\phi=17$ at a sextactic point.]

Ex. 12. Through the crunodes A, B of the quartic of Ex. 8 four conics can be drawn having four-point contact with the curve. Their points of contact are the real sextactic points.

The conic through A, B and three of the sextactic points touches the quartic at one of them.

Ex. 13. If in Ex. 9 the conic $ABPQR$ meets the line joining S to the acnode on the quartic, the conic $ABP'Q'R'$ meets the line joining S' to the acnode on the quartic.

Ex. 14. Obtain the envelope of a line divided in a range of given cross-ratio by a quartic with three biflecnodes.

[Put $f=g=h=0$ in § 3.

The envelope becomes the conic touching the tangents at the biflecnodes, if the range is equianharmonic.]

Ex. 15. The equation of a real quartic with one real and two unreal biflecnodes can be put in the form $xyz^2=(x^2+y^2)^2$.

The quartic can be projected into the lemniscate of Bernouilli

$$r^2=a^2\sin 2\theta.$$

II. Quartic with two Cusps and a Node.

[See § 2. Ex. 10; and Ch. X, Fig. 1.]

Ex. 1. A quartic with two real cusps and a real node can be put in the form

$$y^2z^2+z^2x^2+x^2y^2=2xyz(x+y+mz).$$

The inflexions are unreal or real according as m does or does not lie between 1 and 7.

[The quartic meets $x(y-z)=(t+1)yz$ at

$$(kt^2(kt^2+t+1), \quad kt^2+t+1, \quad kt^2),$$

where

$$2k(m+1)=1.$$

Now using Ch. X, § 1 (v), we get

$$3kt^2+(8k+1)t+3 = 0$$

i.e.

$$9(m+1)(x^2+y^2) = 2(m^2+m+16)xy$$

for the inflexions.]

Ex. 2. The bitangent of the quartic of Ex. 1 is $2x+2y+(m+1)z = 0$. Its points of contact are unreal or real according as m does or does not lie between -1 and 7.

[The points of contact lie on $2x^2-(m-3)xy+2y^2 = 0$. The reader may illustrate by tracing $x^2y^2+x^2+y^2 = 2xy(x+y+8)$, with bitangent $2x+2y+9 = 0$.]

Ex. 3. Find the value of m in Ex. 1, if polygons of 4, 5, or 6 sides can be inscribed in the quartic whose sides touch the curve.

[If t_1, t_2, t_3, t_4 are the parameters of collinear points

$$t_1+t_2+t_3+t_4 = -2(m+1), \qquad t_1^{-1}+t_2^{-1}+t_3^{-1}+t_4^{-1} = -1.$$

Eliminating t_3 and t_4 from these and $t_3 = t_4$, we have

$$4t_1t_2 = \{2(m+1)+(t_1+t_2)\}\,\{t_1t_2+(t_1+t_2)\}.$$

But this is the condition that the tangents at the points $(1, t_1^2, 2t_1)$ and $(1, t_2^2, 2t_2)$ on $z^2 = 4xy$ should meet on

$$4xy = \{2(m+1)x+z\}\,\{y+z\}.$$

Now use the properties of invariants of conics to obtain the condition that polygons of 4, 5, 6 sides can be inscribed in the latter conic and circumscribed to the former. We find

$$m = -2, \quad -\tfrac{5}{4} \text{ or } \tfrac{1}{4}, \quad -\tfrac{11}{10}.]^*$$

Ex. 4. Show that in the case $m = -2$ the cuspidal tangents pass through the points of contact of the bitangent, and that any two lines harmonically conjugate with respect to the tangents at the node meet the curve again at four points such that the tangents at these points form a quadrilateral inscribed in and circumscribed to the quartic.

[(i) The points of contact are $(2, -1, 2)$, $(-1, 2, 2)$.

(ii) The points of contact of the two other tangents to the curve from the point with parameter T are given by

$$2k(T+1)t^2+(kT^2+4kT+T+1)t+2T(kT+1) = 0.$$

If t_1, t_2 are the roots, we readily verify that $x = kt_1^2y$ and $x = kt_2^2y$ form a harmonic pencil with $x^2+y^2 = 2mxy$ in the case

$$m = -2, \qquad k = -\tfrac{1}{2}.]$$

Ex. 5. Show that the line joining the points of contact of the other two tangents from any point of the bitangent of a quartic with a node and two cusps envelops a unicursal curve of the third class.

[The tangents from the point (x, y, z) are given by

$$(t+2)x+kt^3(2kt+1)y-2(kt^2+t+1)^2z = 0.$$

If the point lies on the bitangent $z = -4k(x+y)$, this becomes

$$\{2kt^2+t+2\}\,\{(4k^2t^2+6kt+4k+1)x+k(4kt^2+t^2+6t+4)y\} = 0.$$

Hence if t_1, t_2 are the parameters of the points of contact, while

$$t_1+t_2 = u \quad \text{and} \quad t_1t_2 = v, \qquad 6kv+(8k+1)u+6 = 0.$$

But the line joining the points with parameters t_1, t_2 involves u and v in the third degree.]

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* See Roberts, *Proc. London Math. Soc.*, xxiii (1892), pp. 202–211, for this and the following examples.

Ex. 6. A quartic has cusps A, B and a node C. Show that the following points lie on a conic:

(i) A, B, the points of contact of tangents from A and B, and the points of contact of the bitangent.

(ii) A, B, the inflexions, the remaining intersections of the curve with the tangents at C.

[These and many similar results are proved by noticing that the curve can be projected so as to be symmetrical.]

Ex. 7. A quartic has two cusps A, B and a node C. Show that the line joining C to the intersection O of the tangents at A and B is a double line of the involution determined by the tangents at C and the lines CA, CB.

Given A, B, C, O, find the locus of the inflexions and of the points of contact of the bitangent.

[A quartic with nodes at A, B and a cusp at C touching OA, OB; a conic through ABC. See Ch. V, § 4, Ex. 8.]

Ex. 8. Reciprocate Ex. 5, 6, 7.

III. Quartics with two Flecnodes and a Node.

Ex. 1. The equation of a quartic with two real flecnodes and a node can be put in the form

$$(x^2+y^2)z^2-x^2y^2+2hxy(z+x)(z+y) = 0.$$

Ex. 2. A quartic has two flecnodes A, B and a node C. Show that the following sets of six points lie on a conic:

(i) A, B and the points of contact of the tangents from A, B, C.

(ii) A, B and the intersections (other than A, B, C) with the curve of the tangents at A, B, C.

[These and many similar results follow at once from the fact that the quartic can be projected into a quartic with symmetry.]

Ex. 3. A unicursal quartic with two unreal flecnodes can be projected into the inverse of a conic with respect to the reflection of a focus in the corresponding directrix.

IV. Quartics with a Biflecnode and two Nodes.

Ex. 1. The tangents from A to a quartic with nodes A, B, C form a harmonic pencil with AB, AC, and similarly for the tangents from B. Show that C is a biflecnode; and that the tangents at A form a harmonic pencil with AB, AC, and similarly for B.

Show that the equation of the quartic can be put in one of the forms

$$z^2(x^2 \pm y^2+2mxy) \pm x^2y^2 = 0.$$

[$F = G = 0$ gives $f = g = 0$, if the curve is not degenerate.]

Ex. 2. Show that, if a quartic has nodes A, B and a biflecnode C, it can be projected so as to have two axes of symmetry; unless A is a crunode and B an acnode, or vice versa, when it can be projected so as to have a centre of symmetry at C.

Trace the projected curves in the various cases which can occur.

Prove that, if A and B are acnodes, C is a crunode.

[Put $z = 1$ in Ex. 1.]

Ex. 3. A quartic has crunodes A, B and a biflecnode C. Show that the tangents at A and B meet at the vertices of a quadrangle having A, B, C as diagonal points; and that their remaining intersections with the curve are the vertices of a quadrangle whose diagonal points are C and two points on AB.

[This and similar theorems follow from symmetry, or by putting $f = g = 0$ in §§ 3 and 4.]

Ex. 4. A unicursal quartic cannot have (i) a biflecnode and a cusp, (ii) a biflecnode and a flecnode, (iii) a flecnode and two cusps, (iv) three flecnodes.

§ 7. Unicursal Quartics with Two Distinct Double Points.

Unicursal quartics with only two distinct double points include quartics with a tacnode or rhamphoid cusp and another double point. The reader will readily obtain their properties by modification of the results of Ch. XVIII, §§ 14 and 15. He may illustrate his argument by tracing the curves of Ch. III, § 6 (xi) to (xv), § 8 (v) to (vii).

I. Unicursal Quartics with a Tacnode.*

Ex. 1. A quartic has a tacnode at C and another double point at B; CA is the tangent at C and BA the harmonic conjugate of BC with respect to the tangents at B. Show that its equation is

$$(yz+x^2)^2 = (1-m)\,x^2\,(x^2+p_1xy+p_2y^2)$$

in general.

Show that $p_2 = 0$, if B is a cusp; $(1-m)p_1^2 = 4p_2$ if B is a flecnode; and that B cannot be a biflecnode.

Ex. 2. The bitangents b_1, b_2 of the quartic of Ex. 1 are

$$2z = (1 \pm \sqrt{m})\,(p_1x+p_2y).$$

Ex. 3. If C_1, C_2 are the points of contact of the tangents from C, then C_1, C_2 and the points of contact of b_1 lie on a conic touching the quartic at C.

[$yz+x^2 = (1+\sqrt{m})\,(x^2+p_1xy+p_2y^2)$.]

Ex. 4. The points of contact of b_1 and b_2 lie on a conic touching the quartic at C.

[$2yz+2x^2 = (1-m)\,(2x^2+p_1xy+p_2y^2)$.]

Ex. 5. AB, C_1C_2, b_1, b_2 are all concurrent.

Ex. 6. A conic touches the quartic at C_1, C_2 and touches the tangents from B.

[$(p_1x+p_2y-z)^2 = (1-m)\,p_2\,(x^2+p_1xy+p_2y^2)$.]

Ex. 7. The points of contact of b_1 and its intersections with the tangents from B form an involution with a double point on CA.

Ex. 8. The points of contact of the tangents from A lie on a conic touching the quartic at C.

[Write down the polar cubic of A.]

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* For other examples see *Messenger Math.*, xlvii (1917), p. 95.

II. Unicursal Quartics with a Rhamphoid Cusp.

Ex. 1. A quartic has a rhamphoid cusp at C and another double point at B; CA is the tangent at C and BA the harmonic conjugate of BC with respect to the tangents at B. Show that its equation is

$$(yz+x^2)^2 = 4x^2y\,(x+ay).$$

Show that $a = 0$, if B is a cusp; $a = 1$, if B is a flecnode; and that B cannot be a biflecnode.

Ex. 2. The bitangent b of the quartic of Ex. 1 is $x+ay=z$ touching where $x^2-xy-ay^2=0$.
The tangent from B is $z=(1+a)x$, touching at $B_1\,(1-a,\ 1,\ 1-a^2)$.
The tangent from C is $x+ay=0$, touching at $C_1\,(a,\ -1,\ a^2)$.

Ex. 3. AB, b, CC_1 are concurrent.

Ex. 4. b, CA, B_1C_1 are concurrent.

Ex. 5. B, C, B_1, C_1, and the points of contact of b lie on a conic for which AC and BC_1 are conjugate lines.

[$z(2x+ay) = (2+a)\,x^2+2axy$.]

Ex. 6. If BB_1 meets b at H, the involution pencil formed by CB, CH and by the lines joining C to the points of contact of b has CA as a double ray.

Ex. 7. The points of contact of b and C_1 lie on a conic osculating the quartic at C.

[$yz+x^2 = 2y\,(x+ay)$.]

Ex. 8. The points of contact of the tangents from A lie on a conic osculating the quartic at C.

[Write down the polar cubic of A.]

Ex. 9. If the tangents at B meet the quartic again at F_1 and F_2, the line CA is divided harmonically by F_1F_2 and the tangent at B to the conic touching CA at C and passing through B, F_1, F_2.

[F_1F_2 is $2x+8(a-1)\,y+z=0$, and the conic is $x^2-4xy+2yz=0$.]

Ex. 10. The conic osculating the quartic at C and touching the conic of Ex. 9 at B passes through B_1.

[$yz+x^2 = 2xy$.]

Ex. 11. The conic osculating the quartic at C and passing through the intersections of the curve with AB (other than B) passes through C_1.

[$yz+x^2 = 4y\,(x+ay)$.]

Ex. 12. The conic osculating the quartic at C and passing through the remaining intersections K_1, K_2 of F_1, F_2 with the curve passes through B_1.

[$yz+x^2+2xy = 4\,(1-a)\,y^2$.]

Ex. 13. BK_1 and BK_2 touch the conic of Ex. 12; and K_1K_2 is divided harmonically by CA, CB.

§ 8. Quartics with a Triple Point, &c.

A quartic with a triple point is converted into a unicursal cubic by projecting two points of the quartic into the circular points and then inverting with respect to the triple point.

Of course quadratic transformation with the triple point and two other points on the curve as the points C, A, B of Ch. IX, § 1, comes to the same thing.

Hence from each property of a unicursal cubic can be deduced a property of a quartic with a triple point.

An alternative is to reduce the equation of the quartic to a simple form by a suitable choice of axes or triangle of reference.

The coordinates of any point can be expressed rationally in terms of a parameter by considering the intersection of the curve with a line through the triple point.

If we take the triple point as (0, 0, 1), the equation of the quartic is

$$zu + v = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

where

$$u \equiv a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3,$$

$$v \equiv A_0 x^4 + 4A_1 x^3 y + 6A_2 x^2 y^2 + 4A_3 x y^3 + A_4 y^4.$$

We may simplify this equation by a suitable choice of the other two vertices of the triangle of reference.

For instance, if all three tangents at (0, 0, 1) are real, we may suppose $u \equiv xy\,(x+y)$. Then, replacing z by $z + px + qy$ and choosing p and q so as to make the coefficients of x^3y and xy^3 zero, we may reduce (i) to the form

$$zxy\,(x+y) = ax^4 + 2hx^2y^2 + by^4 \quad . \quad . \quad . \quad . \quad \text{(ii)}.$$

The reader will easily verify that, if

$$(t - \sqrt{a} - \sqrt{b})^2 = a + 2h + b, \quad (T - \sqrt{a} + \sqrt{b})^2 = a + 2h + b,$$

then

$$z + 2t\,(\sqrt{a}\,x + \sqrt{b}\,y) = 0, \quad z + 2T\,(\sqrt{a}\,x - \sqrt{b}\,y) = 0 \quad \text{(iii)}$$

meet the curve (ii) where

$$(\sqrt{a}\,x^2 + txy + \sqrt{b}\,y^2)^2 = 0, \quad (\sqrt{a}\,x^2 + Txy - \sqrt{b}\,y^2)^2 = 0.$$

Therefore the lines (iii) are the four bitangents of (ii), and $z = 0$ is a diagonal of the quadrilateral formed by these bitangents.*

I. Quartics with a Triple Point.

Ex. 1. If O is a triple point of a quartic through A and B, the three conics through A, B, O osculating the quartic at O meet the quartic again on a conic through A, B, O.

[Project A, B to the circular points and invert with respect to O. Then: 'The tangentials of three collinear points of a cubic are collinear.']

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[*] An exception arises if $h^2 - ab$. See Ex. 18, 19, p. 295.

Ex. 2. In Ex. 1 there are three conics through A, B, O which osculate the quartic at P, Q, R. Show that A, B, O, P, Q, R lie on a conic.

Ex. 3. A conic through the triple point O of a quartic meets the curve again in A, B, P, Q, R. If A, B, P are fixed and Q, R vary, OQ and OR trace out an involution pencil.

[See Ch. XIII, § 4, Ex. 12.]

Ex. 4. Two conics through the triple point O of a quartic meet again at A, B, P on the curve, and they both touch the quartic at Q and R respectively. The conic $OABQR$ meets the quartic again at S. Show that OQ and OR are harmonically conjugate with respect to OP and OS, and also with respect to the lines joining O to the other intersections of AB with the quartic.

[See Ch. XIII, § 4, Ex. 13.]

Ex. 5. No triangle can be both inscribed in and circumscribed to a quartic with a triple point.

[The quartic could be transformed quadratically into a quintic with a triple point and three cusps, whose reciprocal would be a quintic with a triple tangent.]

Ex. 6. The line $y = tx$ joining the triple point of § 8 (ii) to the remaining intersections of the curve with the line joining the intersections of $y = t_1 x$ and $y = t_2 x$ with the curve is given by

$$bt^2 + \{b + av - (a + 2h + b)u\} t + av = 0,$$

where $$vt_1 t_2 \equiv 1, \qquad u(1 + t_1)(1 + t_2) \equiv 1.$$

Ex. 7. A triangle is inscribed in a quartic with a triple point O. The lines joining O to the remaining intersections of the quartic with the sides of the triangle form an involution.

[Use Ex. 6.]

Ex. 8. A quartic with a triple point O passes through three fixed points A, B, C and meets BC, CA, AB again at six fixed points. Show that the locus of O is a cubic.

[Use Ex. 7 and Ch. XV, § 3, Ex. 3.]

Ex. 9. If $p = 0$, $q = 0$, $r = 0$, $s = 0$ are four lines, such that $p = s$, $q = s$, $r = s$ are concurrent, $\sqrt{p} + \sqrt{q} + \sqrt{r} + \sqrt{s} = 0$ is a quartic with a triple point and with these lines as bitangents.

[See § 3, Ex. 11.]

Ex. 10. The points of contact of the four bitangents of a quartic with a triple point lie on a conic.

For the quartic § 8 (ii) the equation of the conic is

$$z^2 + 4z(ax + by) - 8(ab + ah)x^2 - 8(ab + bh)y^2 = 0.$$

[It is readily seen that these bitangents are

$$\{z^2 + 4z(ax + by) - 8(ab + ah)x^2 - 8(ab + bh)y^2\}^2 + 64ab(a + 2h + b)\{zxy(x + y) - ax^4 - 2hx^2y^2 - by^4\} = 0;$$

or we may use Ex. 9.]

Ex. 11. Any diagonal of the quadrilateral formed by the bitangents of a quartic with a triple point O meets the curve and the conic through the points of contact of the bitangents in an involution whose double points lie on two tangents at O.

[In § 8 (ii) the diagonal is $z = 0$. The curve

$$16(x-y)(2x+y)y = 12x^4 - 16x^3y - 4x^2y^2 + 8xy^3 + 3y^4$$

with bitangents

$$3x + 2y + 12 = 0, \qquad x + 2 = 0, \qquad x - 4 = 0, \qquad y - 3 = 0,$$

will illustrate Ex. 10 to 15.

The conic $6x^2 + 4xy + 3y^2 - 24x + 8y - 96 = 0$ passes through the points of contact of the bitangents, and the conic

$$52x^2 - 52xy - 27y^2 - 16x - 152y - 48 = 0$$

touches the six inflexional tangents (only two are real). Through the four real intersections of these conics passes another conic through the inflexions.]

Ex. 12. Let E_1, F_1; E_2, F_2; E_3, F_3 be the three pairs of vertices of the quadrilateral formed by the bitangents of a quartic with a triple point O. Show that, if the tangents t_1, t_2, t_3 at O are taken in a certain order, then :

(i) t_2, t_3 are the pair common to the two involutions subtended at O by the points of contact of the pair of bitangents which meets at E_1 and the pair which meets at F_1.

(ii) The two conics through O and the points of contact of these two pairs of bitangents both touch t_1 at O and have four-point contact.

(iii) The conic $OE_2F_2E_3F_3$ touches t_1 at O.

(iv) Through E_2, F_2, E_3, F_3 can be drawn a conic having double contact with the conic through the points of contact of the four bitangents at its intersections with t_1.

(v) Let S_1, S_2, S_3 be the conics of closest contact with the quartic at O, and l_1, l_2, l_3 the lines joining O to their remaining intersections with the quartic. Then prove that t_1 and l_1 divide E_1F_1 harmonically.

(vi) The lines OE_1 and OF_1, t_2 and l_2, t_3 and l_3 form an involution.

(vii) A conic passes through E_1, F_1 and the four intersections of S_2 and S_3. It touches at O the harmonic conjugate of t_1 for t_2 and t_3.

(viii) The conic through the intersections of S_2 and S_3 touching t_1 at O meets t_2 and t_3 in points collinear with the intersection of t_1 and E_1F_1.

[Two pairs of bitangents are

$$z^2 + 4(\sqrt{a} + \sqrt{b})(\sqrt{a}x + \sqrt{b}y)z - 8(h - \sqrt{ab})(\sqrt{a}x + \sqrt{b}y)^2 = 0,$$

$$z^2 + 4(\sqrt{a} - \sqrt{b})(\sqrt{a}x - \sqrt{b}y)z - 8(h + \sqrt{ab})(\sqrt{a}x - \sqrt{b}y)^2 = 0.$$

Add and subtract these equations, and subtract each from the equation of the conic of Ex. 10.

The conics of closest contact at O with § 8 (ii) are

$$zy - a(x^2 - xy + y^2) - 2hy^2 = 0, \qquad zx - b(x^2 - xy + y^2) - 2hx^2 = 0,$$

$$z(x+y) + (3a+b)x^2 + (3a+3b-2h)xy + (a+3b)y^2 = 0.]$$

Ex. 13. A line divided equianharmonically by a quartic with a triple point O envelops a conic touching the six inflexional tangents and touching the Hessian of the tangents at O.

For the curves § 8 (i) and (ii) the tangential equation of the conic is

$$4(A_0A_4 - 4A_1A_3 + 3A_2^2)\nu^2 - 4(a_0A_4 - 3a_1A_3 + 3a_2A_2 - a_3A_1)\nu\lambda$$
$$- 4(a_3A_0 - 3a_2A_1 + 3a_1A_2 - a_0A_3)\nu\mu - 3(a_1a_3 - a_2^2)\lambda^2$$
$$+ 3(a_0a_3 - a_1a_2)\lambda\mu - 3(a_0a_2 - a_1^2)\mu^2 = 0;$$

$$\lambda^2 + \mu^2 - \lambda\mu + 4h(\lambda + \mu)\nu + 4(h^2 + 3ab)\nu^2 = 0.$$

[Eliminate z between $\lambda x + \mu y + \nu z = 0$ and the equation of the quartic, and write down the condition that the resulting pencil of lines is equianharmonic.

The inflexional tangents are divided equianharmonically by any quartic.]

Ex. 14. With the notation of § 8 (i) verify that

$$(3\,C_{20,2}\,z + 2\,C_{11,3})\,(C_{40,0}\,z + C_{31,1}) - 2\,C_{41,0}\,(zu + v)$$
$$\equiv C_{20,2}\,(3\,C_{40,0}\,z^2 + C'_{31,1}\,z + 2\,C_{22,2}),$$

where $C_{20,2}$ &c., are invariants or covariants of u and v given by

$$C_{20,2} \equiv (a_0 a_2 - a_1^2)\,x^2 + \dots,$$
$$C_{11,3} \equiv (a_0 A_2 - 2\,a_1 A_1 + a_2 A_0)\,x^3 + \dots,$$
$$C_{40,0} \equiv a_0^2 a_3^2 - 6\,a_0 a_1 a_2 a_3 + 4\,a_0 a_2^3 + 4\,a_1^3 a_3 - 3\,a_1^2 a_2^2,$$
$$C_{31,1} \equiv \{A_0\,(a_0 a_3^2 - 2\,a_1 a_2 a_3 + a_2^3)$$
$$+ (a_0 a_2 - a_1^2)\,(-4\,A_1 a_3 + 6\,A_2 a_2 - 4\,A_3 a_1 + A_4 a_0)\}\,x + \dots,$$
$$C_{41,0} \equiv A_0\,(a_1 a_3 - a_2^2)^2 + 2\,A_1\,(a_1 a_3 - a_2^2)\,(a_1 a_2 - a_0 a_3)$$
$$+ A_2\,(a_0^2 a_3^2 + 3\,a_1^2 a_2^2 - 2\,a_1^3 a_3 - 2\,a_0 a_2^3)$$
$$+ 2\,A_3\,(a_1 a_2 - a_0 a_3)\,(a_0 a_2 - a_1^2) + A_4\,(a_0 a_2 - a_1^2)^2,$$
$$C'_{31,1} \equiv \{A_0\,(5\,a_0 a_3^2 - 14\,a_1 a_2 a_3 + 9\,a_2^3) + A_1\,(-16\,a_0 a_2 a_3 + 28\,a_1^2 a_3 - 12\,a_1 a_2^2)$$
$$+ A_2\,(30\,a_0 a_2^2 - 12\,a_0 a_1 a_3 - 18\,a_1^2 a_2) + A_3\,(4\,a_0^2 a_3 - 16\,a_0 a_1 a_2 + 12\,a_1^3)$$
$$+ A_4\,(a_0^2 a_2 - a_0 a_1^2)\}\,x + \dots,$$
$$C_{22,2} \equiv \{A_0^2 a_3^2 - 6\,A_0 A_1 a_2 a_3 + A_0 A_2\,(9\,a_2^2 - 2\,a_1 a_3) + 8\,A_1^2 a_1 a_3$$
$$+ A_0 A_3\,(2\,a_0 a_3 - 6\,a_1 a_2) - A_1 A_2\,(12\,a_1 a_2 + 4\,a_0 a_3) + A_0 A_4 a_1^2$$
$$+ 8\,A_1 A_3 a_1^2 + 6\,A_2^2 a_1 a_2 - 2\,A_1 A_4 a_0 a_1 - 4\,A_2 A_3 a_0 a_1 + A_2 A_4 a_0^2\}\,x^2 + \dots.$$

Deduce that the six inflexions lie on the conic

$$3\,C_{40,0}\,z^2 + C'_{31}\,z + 2\,C_{22,2} = 0.$$

For the quartic § 8 (ii) the inflexions lie on

$$2a\,(b + 3h)\,x^2 - 4\,(ab + h^2)\,xy + 2h\,(a + 3h)\,y^2$$
$$= (3a - 2h)\,zx + (3b - 2h)\,zy + z^2.^*$$

[By Ch. VII, § 7, Ex. 14 (v) the inflexions lie on

$$3\,C_{20,2}\,z + 2\,C_{11,3} = 0.]$$

Ex. 15. The three conics of Ex. 10, 13, 14 pass through the same four points.

Ex. 16. Show that the equation of a quartic with a triple point O at which (i) three tangents are real, (ii) one tangent is real, can be put into the form

$$\text{(i)}\quad z\,(3x^2 y - y^3) + x^4 + 6\,ax^2 y^2 + 4\,bxy^3 = 0,$$
$$\text{(ii)}\quad z\,(3x^2 y + y^3) - x^4 + 6\,ax^2 y^2 - 4\,bxy^3 = 0.$$

Show that the bitangents of these curves are

$$z = 2\,tx - \left(\frac{2\,b - t}{3\,t}\right)^2 y,$$

where

$$\text{(i)}\quad 27\,t^4 - (18\,a + 1)\,t^2 + 4\,b^2 = 0,$$
$$\text{(ii)}\quad 27\,t^4 + (18\,a + 1)\,t^2 - 4\,b^2 = 0.$$

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* For this equation, which suggests the more general form, I am indebted to Miss R. E. Colomb.

Ex. 17. The bitangents of a quartic with a triple point O at which the three tangents are real are all real or all unreal.

If one tangent at O is real, two bitangents are real.

Ex. 18. Find the condition that the conic touching the inflexional tangents of the quartics of § 8 (ii) or Ex. 16 should degenerate into a point-pair.

Determine whether this point-pair is real or unreal, and find the bitangents in this case.

[For § 8 (ii) the condition is $h^2 = ab$, and the point-pair is unreal.]

Ex. 19. If the conic touching the inflexional tangents degenerates, three bitangents are concurrent. Each of them is divided harmonically by two of the tangents at the triple point O. The fourth bitangent is divided harmonically by the Hessian of the tangents at O.

Determine whether the points of contact of the bitangents are real or unreal.

[Illustrate by tracing $24\,xy\,(x+y) = (2\,x^2+y^2)^2$, whose bitangents are the line of infinity and

$$2\,x+y+2 = 0, \qquad 2\,x-y+3 = 0, \qquad 2\,x-y-6 = 0.$$

Their points of contact lie on $2\,(x-2)^2+(y-1)^2 = 21.$]

Ex. 20. Determine the condition that the conics of Ex. 10, 14 should degenerate into line-pairs.

[$C_{41,0} = 0$. For § 8 (ii) this becomes $a+h+b = 0$, when the conics become respectively unreal and real line-pairs.]

Ex. 21. By replacing x, y, a, b, h in § 8 (ii) respectively by

$$x+iy, \qquad x-iy, \qquad p+iq, \qquad p-iq, \qquad r-\tfrac{1}{2}(a+b)$$

obtain the equation of a quartic with a triple point at which one tangent is real. Obtain the conics through its inflexions, &c.

Ex. 22. Sketch roughly the different types of quartic with a triple point O, one tangent at O being the line at infinity, and determine the number of bitangents and inflexions for each type.

[There are four such types; one is illustrated by

$$36\,y\,(y+1)\,x = 4\,y^4-84\,y^2+81$$

with bitangents

$$9\,x+10\,y+45 = 0, \qquad 3\,x+4\,y+18 = 0, \qquad 3\,x-2\,y+9 = 0,$$
$$9\,x-8\,y+36 = 0.]$$

II. Quartics having a Triple Point with two Coincident Tangents.

Ex. 1. The equation of a quartic with a triple point at which two tangents coincide may be put in the form $zx^2\,y = y^3(2\,x+y)+ax^4$.

The bitangents are $z+y = \pm 2\sqrt{a}\,(x+y)$.

Ex. 2. The inflexions lie on the conic $3\,yz+2\,xz = 2\,ax^2$.

[Eliminating a between the equations of the curve and its Hessian, we get $2\,y^3+x^2z = 0$, &c.]

Ex. 3. The conic through the points of contact of the bitangents and the triple point O touches the linear branch at O.

[$y\,(y+z) = 2\,ax^2$.]

Ex. 4. The envelope of the line divided equianharmonically by the quartic is a conic touching the inflexional tangents and touching the superlinear branch at O.

[$\mu^2 + 12a\nu^2 = 6\lambda\nu$.]

III. Quartics with a Superlinear Branch of Order Three.

Ex. 1. The equation of a quartic with a superlinear branch of order three can be put in the form $zy^3 = (3ay^2 - x^2)^2$.

The bitangent is $z = 0$.

The inflexions are $(\pm a^{\frac{1}{2}}, 1, 4a^2)$ and the inflexional tangents are

$$z \pm 8a^{\frac{3}{2}}x - 12a^2y = 0.$$

They meet the curve again at $(\mp 3a^{\frac{1}{2}}, 1, 36a^2)$.

Ex. 2. The inflexions are real or unreal according as the points of contact of the bitangent are real or unreal.

Ex. 3. The tangent to the superlinear branch, the bitangent, the line joining the inflexions, and the line joining the remaining intersections of the inflexional tangents with the curve, are all concurrent.

[Putting $z = 1$, we see that the curve may be projected into one with an axis of symmetry. Trace this projection, distinguishing the cases

$$a > 0, \qquad a = 0, \qquad a < 0.]$$

Ex. 4. Any line met by the quartic in an equianharmonic range passes through the intersection of the inflexional tangents.

Ex. 5. The line divided harmonically by the quartic envelops a conic touching the curve at the singular point, and touching the inflexional tangents.

[It is $\lambda^2 + 16a\mu\nu + 128a^3\nu^2 = 0$.]

Ex. 6. A conic passes through the inflexions and the remaining intersections of the inflexional tangents with the curve, which touches the bitangent and touches the quartic at the singular point.

[$yz = 4ax^2$.]

IV. Unicursal Quartics of Class Six with one distinct Double Point.

Ex. 1. A quartic has two linear branches having three-point contact with one another. Show that its equation can be put in the form

$$(yz + x^2)^2 = y^2(px^2 + qy^2).$$

[By Ch. III, § 8, Ex. 6, the equation is

$$0 = (yz + x^2)^2 + 2y(yz + x^2)(lx + my) + y^2(ax^2 + 2hxy + by^2).$$

Choose a triangle of reference ABC such that C is the double point, B the other intersection of the osculating conics at C, A the intersection of the tangent at C with the other common tangent of the conics. We find that then $l = m = h = 0$.]

Ex. 2. Show how to transform the quartic into a conic by quadratic transformation.

[Put $z - x^2/y$ for z in Ex. 1.]

Ex. 3. The bitangents of the quartic of Ex. 1 are

$$(p^2 + 4q)y = 4pz \qquad \text{and} \qquad y^3 = 0.$$

Ex. 4. Show that the quartic can be projected into one with an axis of symmetry. Indicate roughly its shape in the cases

$$p > 0,\ q > 0; \qquad p > 0,\ q < 0; \qquad p < 0,\ q > 0.$$

Put $z = 1$. Properties of the curve can be written down from the symmetry; e.g. 'A conic through two inflexions and the points of contact of a bitangent touches the curve at C.']

Ex. 5. Express the coordinates of any point of the curve rationally in terms of a parameter.

[Put $2tp^{\frac{1}{2}}x = (1-t^2)\,q^{\frac{1}{2}}y$.]

V. Unicursal Quartics of Class Five with one distinct Double Point.

Ex. 1. If in IV, Ex. 1, $a = l^2$, the equation can be put in the form

$$(yz + x^2)^2 = xy^3.$$

[Replacing z by $z - lx - my$ we reduce the equation to

$$(yz + x^2)^2 = y^3(px + qy).$$

An infinite number of conics meet the curve at seven points coinciding with $(0, 0, 1)$, but no conic meets it eight times there.

Choose a new triangle of reference ABC such that C is the singularity, CA the tangent at C, B the point of contact of the tangent from C, AB the tangent at B to that conic of closest contact at C which goes through B.]

Ex. 2. Any point on the quartic is $(t^2, 1, -t^4 - t)$.

If the points with parameters t_1, t_2, t_3, t_4 are collinear,

$$t_1 + t_2 + t_3 + t_4 = 0, \qquad t_2t_3t_4 + t_1t_3t_4 + t_1t_2t_4 + t_1t_2t_3 + 1 = 0.$$

Ex. 3. The tangent at any point is

$$(1 + 4t^3)\,x + t^2(1 - 2t^3)\,y + 2tz = 0,$$

and meets the curve again at the points with parameters $\pm(1/2t)^{\frac{1}{2}} - t$.

The curve is of class 5.

Ex. 4. One of the inflexions is real and two unreal. They lie on a conic having double contact at B and C with that conic which passes through B and has closest contact at C with the quartic.

The inflexional tangents meet the curve again at points lying on such a conic.

[$t = \frac{1}{2}$, $\frac{1}{2}\omega$, $\frac{1}{2}\omega^2$. Conics are $yz + 9x^2 = 0$, $8yz + 35x^2 = 0$. More generally, if any such conic meets the curve at P, Q, R, the tangents at P, Q, R meet the quartic again in six points lying by threes on two such conics.]

Ex. 5. The bitangents coincide with $y = 0$.

Ex. 6. Derive properties of the quartic from those of the conic by quadratic transformation.

[Putting $z - x^2/y$ for z we obtain $z^2 = xy$.]

CHAPTER XVIII

QUARTICS OF DEFICIENCY ONE OR TWO

§ 1. Nodal Quartics.

In this chapter we shall consider quartics with deficiency one or two.

A quartic with deficiency two has a single node or cusp. If it has a node O, we take this node as $(0, 0, 1)$. Each of the tangents at O meets the curve at another point. We call these points I and J, and take the line IJ as $z = 0$.

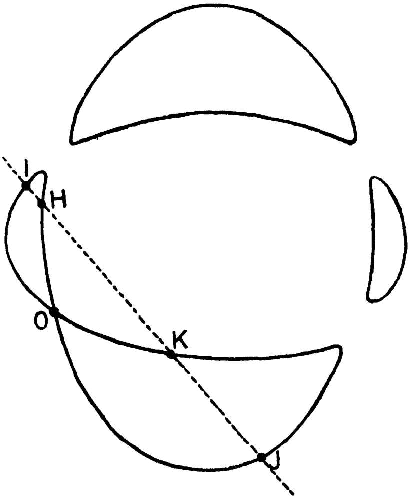

Fig. 1.
$100(2x^2+y^2-16x-4y)(x^2+3y^2-8x-12y)+(4x+y)(2x+3y)=0.$

Since $(0, 0, 1)$ is a node, the equation of the quartic is of the form

$$u_4+2u_3z+u_2z^2=0\,;$$

where u_2, u_3, u_4 are homogeneous of degree 2, 3, 4 in x and y, and $u_2=0$ is the equation of the tangents at the node. Also, since $z=0$ and $u_2=0$ intersect at two points on the curve,

u_2 is a factor of u_4. Hence the equation of the quartic may be written

$$u_2v_2+2u_3z+u_2z^2=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

$v_2=0$ being the equation of the lines joining O to the points H and K at which IJ meets the quartic again (Figs. 1 and 2).

The six points of contact of the tangents to the quartic from the node O lie on a conic touching OH and OK at H and K.

For the polar cubic of O is $u_3+u_2z=0$, which meets the quartic six times at O and at the six intersections of the quartic with $z^2=v_2$, as is seen by writing (i) in the form

$$u_2(v_2-z^2)+2z(u_3+u_2z)=0.$$

Ex. 1. We may without loss of generality take u_2 as xy or x^2-y^2 if the node is a crunode, or as x^2+y^2 if the node is an acnode.

Ex. 2. A line meets the quartic in A, B, C, D and OA, OB, OC, OD meet the quartic again in A', B', C', D'. Show that O, H, K, A', B', C', D' lie on a conic.

[If the line is $z=v_1$, the lines OA, OB, OC, OD are

$$u_2v_2+2v_1u_3+u_2v_1^2=0,$$

and the required conic is $v_2=v_1z$.]

Ex. 3. A line through H meets the quartic again in A, B, C; and OA, OB, OC meet the quartic again in A', B', C'. Show that K, A', B', C', are collinear, and that the lines $HABC$, $KA'B'C'$ meet on the conic through the points of contact of the tangents from O.

Ex. 4. If H and K are unreal, the quartic can be projected so as to be its own inverse with respect to a circle j whose centre is the node O. Eight foci of the projected 4-ic lie on j, and the 4-ic is the envelope of a circle cutting j orthogonally whose centre lies on the polar reciprocal with respect to j of the polar cubic of O. The foci of the 4-ic on j are the points of contact with j of the common tangents of j and the polar cubic of O.

Conversely, the envelope of a circle cutting any given circle orthogonally whose centre lies on a tricuspidal 4-ic with an infinite bitangent is a nodal 4-ic.

[Project H and K into the circular points. See also Ch. XI, § 11, Ex. 3.]

Ex. 5. If IJ touches the quartic at H, the points of contact of tangents from O lie on two lines through H.

Ex. 6. Show that:

(i) The equation of a quartic with a biflecnode O can be put in the form $xyz^2+u=0$ or $(x^2+y^2)z^2+u=0$, where u is homogeneous of degree 4 in x and y.

(ii) The points of contact of the tangents from O are collinear.

(iii) The sixteen inflexions other that O lie on another quartic.

(iv) Any line through O is divided harmonically by the quartic and a fixed line.

[(iii) Combine the equations of the curve and its Hessian.

(iv) Putting $z=1$, we see that the curve can be projected so as to have O as a centre of symmetry. From this fact other properties of a biflecnodal quartic may be written down.]

Ex. 7. Show that:

(i) The equation of a quartic with a flecnode O can be put in the form

$$xyz^2+2yuz+x^2v=0,$$

where u and v are homogeneous of degree 2 in x and y.

(ii) The tangents from O to the curve lie on a conic touching the non-inflexional branch at O.

(iii) If a line meets the quartic in A, B, C, D and OA, OB, OC, OD meet the quartic again in A', B', C', D', then A', B', C', D' lie on a conic touching the non-inflexional branch at O.

[(ii) $xz+u=0$, (iii) $x(z+w)+2u=0$, if the line is $z=w$.]

Ex. 8. Show that the theorem of § 1 can be generalized as follows: Given an n-ic with an $(n-2)$-ple point O, we can find an r-ic with an $(r-2)$-ple point at O such that any line through O is divided harmonically by the curves; r being any given number $\geqslant \frac{1}{2}n$. Each curve passes through the points of contact of the tangents from O to the other.]

[If $u_{n-2}z^2+2u_{n-1}z+u_n=0$, $v_{r-2}z^2+2v_{r-1}z+v_r=0$ are the curves,

$$u_{n-2}v_r+u_n v_{r-2}\equiv 2u_{n-1}v_{r-1}.$$

See Bateman, *Archiv der Math. und Physik*, xiii (1908), p. 48.]

Ex. 9. Show that Ex. 4 can be generalized as follows: If an n-ic with an $(n-2)$-ple point O is self-inverse with respect to a circle j with centre O, it is the envelope of a circle cutting j orthogonally whose centre lies on the polar reciprocal with respect to j of the first polar curve of O. This polar reciprocal passes through $4(n-2)$ foci of the n-ic lying on j.

Ex. 10. A large number of polished wires in the form of concentric circles lie on a table. Light emanating from a fixed point is reflected at the wires to another fixed point. Find the locus of the point of reflexion.

[A nodal circular quartic.]

§ 2. Cuspidal Quartics.

If a quartic has a single cusp O, we may take O as $(0, 0, 1)$. As in § 1 the equation of the quartic is

$$u_4+2u_3z+u_2z^2=0,$$

where in this case u_2 is a perfect square. Most of the results of § 1 hold with slight modifications. The points I, J coincide at the intersection of the quartic with the cuspidal tangent, and the tangent at this intersection meets the quartic again in H and K.

We shall leave the verification of these facts as an exercise to the reader, and give here only the method of finding the bitangents of a cuspidal quartic.*

The polar cubic of the cusp O has also a cusp at O and the cuspidal tangents of the cubic and quartic coincide; for this cubic is

$$U\equiv zu_2+u_3=0.$$

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* H. A. Richmond, *Quarterly Journal Math.*, xxvi (1893), pp. 5–26.

The quartic is

$$U^2 = u_3^2 - u_2 u_4.$$

Now by a proper choice of the triangle of reference we may reduce the equation of the polar cubic to

$$zx^2 + y^3 = 0.$$

We have then $u_2 = x^2$, $u_3 = y^3$, and the quartic becomes

$$(zx^2 + y^3)^2 = y^6 - x^2 u_4.$$

Suppose that

$$y^6 - x^2 u_4 \equiv (y+ax)(y+bx)(y+cx)(y+dx)(y+ex)(y+fx);$$

where a, b, c, d, e, f are subject only to the relation

$$a+b+c+d+e+f = 0.$$

We shall denote $y+ax$, $y+bx$, ... by $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, and zx^2+y^3 by U.

Now

$$2U = \alpha\beta\gamma + \delta\epsilon\zeta$$

is equivalent to

$$x^2\{2z - (bc+ca+ab+ef+fd+de)\,y - (abc+def)\,x\} = 0,$$

and meets the quartic

$$U^2 = \alpha\beta\gamma\delta\epsilon\zeta$$

where

$$(\alpha\beta\gamma - \delta\epsilon\zeta)^2 = 0.$$

Hence

$$2z - (bc+ca+ab+ef+fd+de)\,y - (abc+def)\,x = 0 \quad . \tag{i}$$

and the nine similar equations are the ten bitangents of the quartic.*

Writing the quartic in the form

$$\begin{aligned}(2z - p_3x - p_2y)^2 x^2 &+ 4(2z - p_3x - p_2y)\,y^3 \\ &= \{(4p_6 - p_3^2)\,x^2 + 2(2p_5 - p_2p_3)\,xy + (4p_4 - p_2^2)\,y^2\}\,x^2,\end{aligned}$$

where p_2, p_3, p_4, p_5, p_6 are the sum of the products of a, b, c, d, e, f two, three, four, five, six at a time, we see that the tangent at $I\,(\equiv J)$ is

$$2z - p_3x - p_2y = 0,$$

and that the points H, K of § 1 are its intersections with

$$(4p_6 - p_3^2)\,x^2 + 2(2p_5 - p_2p_3)\,xy + (4p_4 - p_2^2)\,y^2 = 0.$$

Some of the properties of the cuspidal quartic may be verified on Fig. 2. In it $\alpha, \beta, \ldots$ denote the points of contact of $\alpha = 0$, $\beta = 0$, ... They are the points with abscissae $-\frac{1}{5}, -1, -\frac{1}{2}, \frac{3}{5}, \frac{3}{4}, 5$; the cusp being $(0, \infty)$, and the cuspidal tangent $x = 0$.

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* The six quantities $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ can be divided into two sets of three in ten ways.

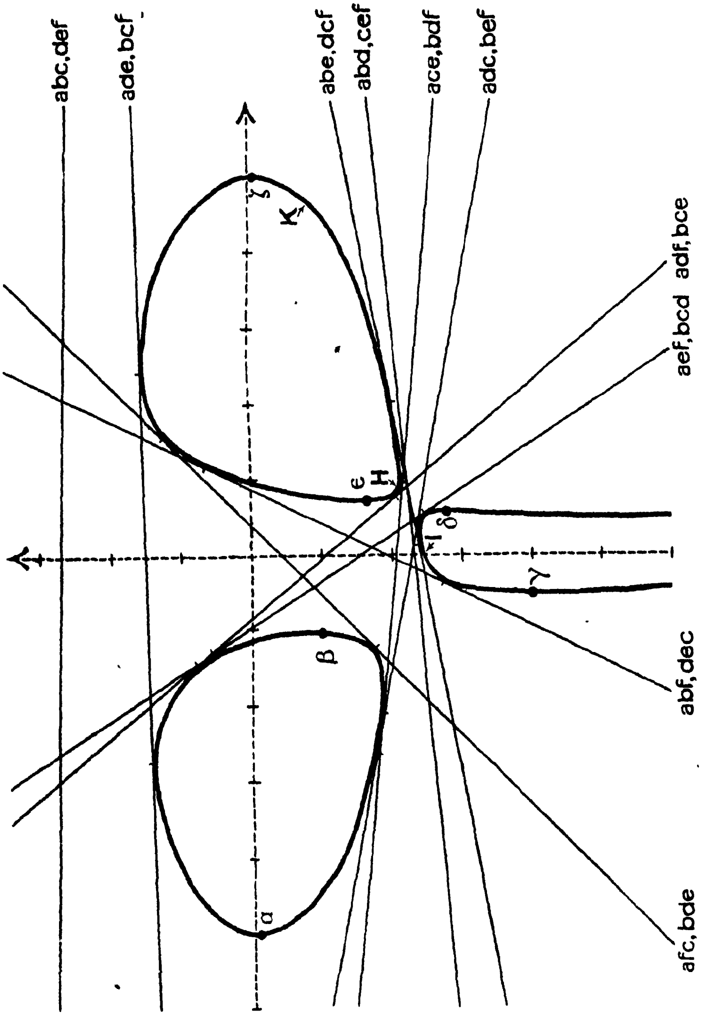

Fig. 2.

$225(yx^2+1)^2 = (5+x)(1+x)(1+2x)(3-5x)(3-4x)(5-x)$.

The bitangent (i) may be conveniently denoted by the symbol (abc, def). In Fig. 2 the equations of the bitangents are as follows:

$$(abc,\ def)\ x+45y=122;\ \ (abd,\ cef)\ x-10y=23;$$
$$(abe,\ dcf)\ x-5y=12;\ \ (abf,\ dec)\ 991x-450y=859;$$
$$(adc,\ bef)\ 3x+15y=-34;\ \ (aec,\ bdf)\ 3x+30y=-61;$$
$$(afc,\ bde)\ 241x-225y=92;\ \ (ade,\ bcf)\ x-45y=-68;$$
$$(adf,\ bce)\ 65x+50y=-51;\ \ (aef,\ bcd)\ 41x+25y=-38.$$

The bitangents (abc, def) and (abd, cef) have unreal points of contact.

Ex. 1. Of the ten bitangents of a cuspidal quartic, ten, four, or two are real.

[If a, b, c, d, e, f are real, ten bitangents are real. If four or none of them are real, four bitangents are real. If two of them are real, two bitangents are real.]

Ex. 2. No two of a, b, c, d, e, f can be equal unless the quartic has a second double point.

[If $a = b$, $(1,\ -a,\ a^3)$ is a node.]

Ex. 3. Through the points of contact of any bitangent can be drawn two conics touching the quartic at the cusp and each passing through the points of contact of three tangents from the cusp.

[$xz = abc\,x^2 + (bc+ca+ab)\,xy + (a+b+c)\,y^2$, &c.]

Ex. 4. Show that the points of contact of $\epsilon = 0$, $\zeta = 0$ and the bitangents (abe, cdf) and (abf, cde) all lie on a conic through the cusp.

[$U(\epsilon+\zeta) = (\alpha\beta+\gamma\delta)\,\epsilon\zeta$.]

Ex. 5. The point of contact of $\alpha = 0$, the intersection of $\beta = 0$ with the bitangent (acd, bef), and the intersections of the bitangents (abe, cdf) and (abf, cde) are collinear.

[They lie on $2U+\alpha^2\beta = \alpha\gamma\delta+\alpha\beta\epsilon+\alpha\beta\zeta$, which is a straight line. The reader may refer to Richmond, *loc. cit.*, for further examples.]

Ex. 6. What modifications must be made in § 1, Ex. 4, in the case of a cuspidal cubic?

[Six foci lie on j. The centre locus is a cuspidal cubic having three-point contact with the line at infinity.]

Ex. 7. A quartic with deficiency two is referred to a triangle both inscribed and circumscribed to the curve, the double point being taken as (1, 1, 1). Show that its equation takes the form

$$ayz\,u^2 + bzx\,v^2 + cxy\,w^2 = 0,$$

the tangents at the double point being $au^2+bv^2+cw^2 = 0$; where

$$u \equiv (1+A)\,x-y-Az,\quad v \equiv -Bx+(1+B)\,y-z,\quad w \equiv -x-Cy+(1+C)\,z.$$

Show also that $\alpha u+\beta v+\gamma w \equiv 0$, and that the quartic has a cusp if

$$bc\,\alpha^2 + ca\,\beta^2 + ab\,\gamma^2 = 0;$$

where α, β, γ denote $BC+B+1$, $CA+C+1$, $AB+A+1$.

Show that the polar cubic of the node is

$$a\,(y+z)\,u^2 + b\,(z+x)\,v^2 + c\,(x+y)\,w^2 = 0.$$

§ 3. Bicircular Quartics.

If a real quartic has a pair of unreal nodes, they may be projected into the circular points at infinity. The quartic then becomes bicircular, and its equation is of the form

$$c(x^2+y^2)^2+2(lx+my)(x^2+y^2)$$
$$+ax^2+2hxy+by^2+2gx+2fy=0,$$

if the origin is taken on the curve.

If we now invert with respect to a circle with centre the origin and unit radius, the quartic becomes the circular cubic

$$c+2(lx+my)+ax^2+2hxy+by^2+2(gx+fy)(x^2+y^2)=0.$$

Since foci invert into foci and a circle and two inverse points into a circle and two inverse points, the properties of the foci of a circular cubic proved in Ch. XIV, §§ 2, 3, hold for a bicircular quartic, namely :—

A bicircular quartic is self-inverse with respect to each of four mutually orthogonal circles each of which passes through four foci. If the four real foci are concyclic, the quartic consists of two ovals and three of the four circles are real. If the four real foci are not concyclic, the quartic consists of a single oval and two of the circles are real, while each passes through two real foci.

The reader will at once verify that, if the four lines

$$y^2+(x\pm\alpha)^2=0$$

each meet a bicircular quartic at only one finite point, the coefficients of x^3, x^2y, xy^2, y^3, xy in the equation of the quartic are all zero. Hence the equation of a bicircular quartic becomes

$$(x^2+y^2)^2+ax^2+by^2+2gx+2fy+c=0 \quad . \quad . \quad . \quad \text{(i)}$$

when the line joining a pair of singular foci is taken as $y=0$, and the middle point O of the line is taken as origin. The four singular foci of this quartic are readily shown to be $(\pm\frac{1}{2}\sqrt{(b-a)}, 0)$, $(0, \pm\frac{1}{2}\sqrt{(a-b)})$. We shall suppose $b>a$ in the following.

The quartic (i) may be written

$$(x^2+y^2-t)^2+(a+2t)x^2+(b+2t)y^2+2gx+2fy+c-t^2=0.$$

Hence the conic

$$(a+2t)x^2+(b+2t)y^2+2gx+2fy+c-t^2=0 \quad . \quad . \quad \text{(ii)}$$

has contact with the quartic at four points lying on a circle

$$x^2+y^2=t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

with centre O, whatever may be the value of t. If

$$4t^4+2(a+b)t^3+(ab-4c)t^2-2(ac+bc-f^2-g^2)t$$
$$-(abc-af^2-bg^2)=0 \quad . \quad . \quad \text{(iv),}$$

the conic (ii) is a line-pair. It is therefore a pair of bitangents in this case.

We may exhibit the bicircular quartic (i) as an envelope in another manner. In fact it is the envelope of a circle whose centre lies on the conic

$$4x^2/(a+2t)+4y^2/(b+2t)+1=0 \quad . \quad . \quad . \quad \text{(v)}$$

and which cuts orthogonally the circle

$$x^2+y^2+2gx/(a+2t)+2fy/(b+2t)+t=0 \quad . \quad . \quad \text{(vi)},$$

where t is any root of equation (iv).

For, if we take the centre of the variable circle at

$$(\tfrac{1}{2}(-a-2t)^{\frac{1}{2}}\cos\phi,\ \tfrac{1}{2}(-b-2t)^{\frac{1}{2}}\sin\phi) \quad . \quad . \quad \text{(vii)},$$

its equation is

$$\begin{aligned}(x^2+y^2-t) &= \cos\phi\,\{(-a-2t)^{\frac{1}{2}}x-(-a-2t)^{-\frac{1}{2}}g\} \\ &\quad +\sin\phi\,\{(-b-2t)^{\frac{1}{2}}y-(-b-2t)^{-\frac{1}{2}}f\} \quad . \quad . \quad \text{(viii)},\end{aligned}$$

whose envelope is found in the usual manner to be (i), on making use of (iv).

The variable circle has double contact with its envelope (i).*

The foci of (v) are the singular foci of (i).

The curve (i) is self-inverse with respect to the circle (vi), since the variable circle is self-inverse with respect to (vi), being orthogonal to it. Hence the four circles with respect to which (i) is self-inverse are obtained by putting into equation (vi) any value of t derived from (iv).

The ordinary foci of the quartic (i) are the intersections of (v) and (vi), each value of t giving four such foci.

For, if P is any intersection, the circular lines through P form a degenerate circle orthogonal to (vi) and touching the envelope of the circle (viii), which is (i).

The conics (v), where t is a root of (iv), are called the 'focal' or 'deferent' conics of the bicircular quartic.

The reader will notice that (ii) and (vi) have the same centre. It follows that a pair of bitangents passes through the centre of each circle for which the quartic is self-inverse, as is geometrically obvious.

In Fig. 3 is shown the bicircular quartic, whose real foci $(0, 1)$, $(-\frac{21}{25}, -\frac{24}{25})$, $(\frac{12}{5}, -\frac{3}{5})$, $(\frac{9}{5}, \frac{4}{5})$ lie on the focal conic $x^2+9y^2=9$. The centres of the circles for which the quartic is self-inverse are $(\frac{6}{7}, 3)$, $(\frac{42}{5}, \frac{1}{15})$, $(\frac{21}{20}, \frac{3}{10})$, $(\frac{56}{75}, -\frac{12}{25})$, being the harmonic points of the quadrangle formed by the real foci

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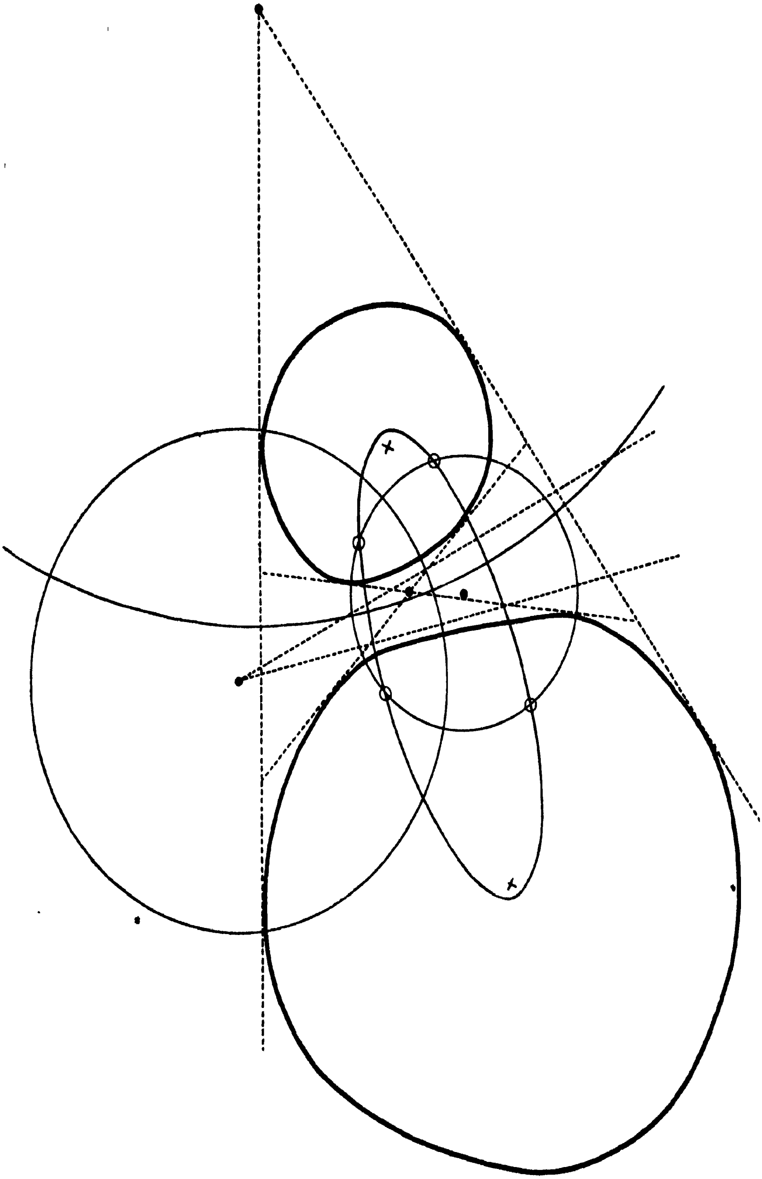
* As is the case in general with any singly infinite family of circles. For the similar result for circular cubics see Ch. XIV, § 3.



and the centre of the circle through the foci. The quartic is drawn by the method described in Ch. XI, § 11, Ex. 3.

The real foci are denoted by ○, the singular foci by ×, and the centres by • in the figure.

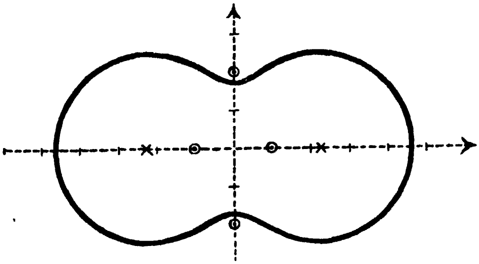


Fig. 4.

$(x^2+y^2)^2-22x^2-2y^2-4=0.$

Singular foci $(\pm\sqrt{5}, 0)$; ordinary foci $(\pm 1, 0)$, $(0, \pm 2)$.

Ex. 1. The inverse of a bicircular quartic with respect to a point not on the curve is a bicircular quartic; the inverse with respect to a point on the curve is a circular cubic.

Ex. 2. Through any point O of a bicircular quartic three real circles of curvature pass besides the circle of curvature at O, and the three points of osculation lie on a circle through O.

[Invert with respect to O. We may derive other theorems by inverting properties of a circular cubic, e.g. Ch. XIV, § 3, Ex. 10, 11.]

Ex. 3. The circles of curvature of a bicircular quartic at its four intersections with any circle for which it is self-inverse have four point contact.

Ex. 4. If O is the point half-way between the real singular foci of a bicircular quartic, show that:

(i) O is half-way between the unreal singular foci, and the sum of the squares of the distances of O from the singular foci is zero.

(ii) The points of contact of the tangents from O lie on a conic.

(iii) O is equidistant from the middle points of AB and CD, where A, B, C, D are the intersections of any line with the quartic.

[Use equation (i) of § 3, and write down the first polar of O.]

Ex. 5. The bisectors of the angle between any pair of bitangents of a bicircular quartic are parallel and perpendicular to the line joining the real singular foci.

[Equation (ii) of § 3 has no term in xy.]

Ex. 6. The four intersections of two pairs of bitangents lie on a circle whose centre O is half-way between the real singular foci.

The sum of the squares of the radii of the circles obtained by taking two pairs of bitangents and also the other two pairs is the same for each way of dividing into pairs.

Ex. 7. The eight points of contact of any two of the conics (ii) of § 3 lie on a conic.

The eight points of contact of any two pairs of bitangents lie on a conic.

[The points of contact of the conics for which $t = t_1$ and $t = t_2$ lie on

$$(a+2t_1)x^2+(b+2t_1)y^2+2gx+2fy+c-t_1^2+(t_2-t_1)(x^2+y^2-t_1) = 0,$$

i.e. $$ax^2+by^2+2gx+2fy+c+(t_1+t_2)(x^2+y^2)-t_1t_2 = 0.]$$

Ex. 8. The four intersections of any two of the conics (ii) of § 3 lie on a circle with centre O.

The rectangular hyperbola through the intersections has fixed asymptotes.

The conic through O and the intersections has a fixed tangent at O.

[Note the case in which the conics are pairs of bitangents as in Ex. 6.]

Ex. 9. Find the centres of the circles for which the quartic (i) of § 3 is self-inverse; and show that they lie on a rectangular hyperbola through O whose asymptotes are parallel and perpendicular to the line joining the real singular foci.

[$(-g/(a+2t),\ -f/(b+2t))$, where t is given by (iv).

The centre of (ii) lies on $b-a = g/x-f/y$.]

Ex. 10. The centroid of the four finite intersections of a bicircular quartic with any circle whose centre O is half-way between the real singular foci of the quartic is a fixed point V. It is also the centroid of any four concyclic foci, and the centroid of the four points of contact of any pair of bitangents. It is the intersection of the axes of the two parabolas of the family (ii); and the centre of the rectangular hyperbola of Ex. 9.

[Eliminate x or y between (i) and (iii) or between (v) and (vi). V is the point $(g/(b-a),\ f/(a-b))$.]

Ex. 11. In Ex. 10 the ratio of the distances from $x = 0$ of the centre of (ii) or (vi) and of V is the square of the eccentricity of (v).

[This enables us to determine the focal conic corresponding to any circle for which the quartic is self-inverse]

Ex. 12. The centres of three circles for which a bicircular quartic is self-inverse are the vertices of the common self-conjugate triangle of the fourth circle and its focal conic.

[They are the harmonic points of the quadrangle whose vertices are the four foci on the fourth circle.]

Ex. 13. If a bicircular quartic consists of two ovals one inside the other, the focal conic through the real foci is a hyperbola and the other three focal conics are ellipses.

If the quartic consists of two ovals external to each other, the focal conic through the real foci is an ellipse and the other three focal conics are hyperbolas.

If the quartic consists of one oval, one focal conic is an ellipse, one is a hyperbola, and two are unreal.

[Use Ex. 11 and Ch. XI, § 11, Ex. 3, 4.]

Ex. 14. The directrices corresponding to four concyclic foci pass through the centre of the circle and form a pencil of the same cross-ratio as that subtended by the foci at O.

[The line joining O to the point (vii) forms a pencil homographic with that traced out by the chord of contact of (viii) with its envelope.]

Ex. 15. The normals to a bicircular quartic at the points where it meets a circle for which it is self-inverse touch a conic whose foci are the singular foci of the quartic.

[The corresponding focal conic. It follows that these normals form a quadrangle whose opposite sides subtend supplementary angles at either singular focus, &c., &c.]

Ex. 16. The pair of bitangents through the centre of any one of the circles for which a bicircular quartic is self-inverse are perpendicular to the asymptotes of the corresponding focal conic.

[See Fig. 3 and compare Ex. 13.]

Ex. 17. The distances of any point P on a two-circuited bicircular quartic from three real foci A, B, C are connected by a linear relation $l\,.\,PA+m\,.\,PB+n\,.\,PC=0$, where l, m, n are constants.

[Invert Ch. XIV, § 3, Ex. 20, with respect to any point, and we find that the result of Ch. XIV, § 3, Ex. 23, holds for the bicircular quartic.
As in Ch. XIV, § 3, Ex. 26, we obtain results for the one-circuited quartic.
As an alternative, we may invert the result of § 6 (iv) or § 10, Ex. 3.]

Ex. 18. If a two-circuited bicircular quartic with real foci A, B, C passes through real points P, Q, R, then a two-circuited bicircular quartic with foci P, Q, R passes through A, B, C.

[By Ex. 17 we have $\begin{vmatrix} PA & PB & PC \\ QA & QB & QC \\ RA & RB & RC \end{vmatrix}=0$; which is unaltered by interchange of P, Q, R and A, B, C.]

Ex. 19. Any bicircular quartic can be inverted into a bicircular quartic symmetrical about two perpendicular lines on which lie the real foci and singular foci.

[Invert with respect to the intersection of two real circles for which the quartic is self-inverse and on which lie the real foci. See Fig. 4.]

Ex. 20. Obtain the equations of the bicircular quartic symmetrical about the axes of reference whose real singular and ordinary foci are given and are on these axes.

[There are three types. We may take the real singular foci as $(\pm m, 0)$. Then the curve is

$$(x^2+y^2)^2-2(l+m^2)x^2-2(l-m^2)y^2+k\alpha^2\beta^2=0,$$

where

(1) The real foci are $(\pm\alpha, 0)$, $(\pm\beta, 0)$, $l^2=(m^2-\alpha^2)(m^2-\beta^2)$, $k=+1$.
(2) The real foci are $(0, \pm\alpha)$, $(0, \pm\beta)$, $l^2=(m^2+\alpha^2)(m^2+\beta^2)$, $k=+1$.
(3) The real foci are $(\pm\alpha, 0)$, $(0, \pm\beta)$, $l^2=(m^2-\alpha^2)(m^2+\beta^2)$, $k=-1$.

As an exercise the reader may discuss the relative position of singular and ordinary foci, and the nature of the ovals and bitangents for each type. He may also find the relation connecting the distances of any point of the curve from three foci (Ex. 17). For this purpose he may put $m^2=\alpha^2\cosh^2\epsilon-\beta^2\sinh^2\epsilon$ in type (1).]

Ex. 21. Find the locus of the singular foci of a two-circuited bicircular quartic, given the real ordinary foci.

[The circular cubics with the given foci. See Ch. XIV, § 3, Ex. 27.]

Ex. 22. Show that the equation of a two-circuited bicircular quartic can be put in the form

$$(x^2+y^2)^2-2\,(lx+my)\,(x^2+y^2-1)+ax^2+2\,lmxy+by^2+1=0.$$

[Take $x^2+y^2+1=0$ as the unreal circle with respect to which the quartic is self-inverse, and the axes of reference parallel and perpendicular to the line joining the real singular foci.]

Ex. 23. Obtain the equations of the conics touching the quartic of Ex. 22 in four points, the equations of the bitangents, and the equations of the circles with respect to which the quartic is self-inverse.

[The quartic is $u^2+v=0$, where

$$u \equiv x^2+y^2-lx-my+t,$$

$$v \equiv (a-l^2-2t)\,x^2+(b-m^2-2t)\,y^2+2\,(t+1)\,(lx+my)+(1-t^2).$$

The conics are $v=0$; and the bitangents are $v=0$, where

$$t=-1 \text{ or } t_1,\, t_2,\, t_3,$$

which are the roots of the equation

$$4t^3-2\,(a+b+2)\,t^2+(ab+2a+2b-4l^2-4m^2-l^2m^2)\,t$$
$$+(2am^2+2bl^2-ab-3l^2m^2)=0.$$

The required circles have their centres at the centre of $v=0$ where $t=-1,\ t_1,\ t_2,\ t_3$, and are mutually orthogonal. Their equations are therefore $x^2+y^2+1=0$, $S_1=0$, $S_2=0$, $S_3=0$, where

$$S_1 \equiv x^2+y^2-2\,(t_1+1)\,lx/(2t_1+l^2-a)-2\,(t_1+1)\,my/(2t_1+m^2-b)-1, \text{ \&c.}]$$

Ex. 24. Show that the squares of the tangents from any point of a two-circuited bicircular quartic to the three real circles with respect to which the quartic is self-inverse are connected by a homogeneous linear relation.

Conversely, if the squares of the tangents from P to three given circles are connected by a homogeneous linear relation, the locus of P is a bicircular quartic; which is self-inverse with respect to each of the given circles, if these circles are mutually orthogonal.

[The quartic of Ex. 22 is

$$(t_2-t_3)\,(2t_1+l^2-a)\,(2t_1+m^2-b)\,S_1^2+\ldots+\ldots=0.]$$

Ex. 25. Show how to find the foci of $aS_1^2+bS_2^2+cS_3^2=0$, where $S_1=0$, $S_2=0$, $S_3=0$ are given mutually orthogonal circles.

[The quartic is the envelope of $\lambda S_1+\mu S_2+\nu S_3=0$ where

$$\lambda^2/a+\mu^2/b+\nu^2/c=0.$$

Now make the radius of this circle zero.]

Ex. 26. Given the circles with respect to which a bicircular quartic is self-inverse and one point on the curve, find the locus of the singular foci.

Ex. 27. Find the locus of P, if

$$\text{(i)}\quad PA\,.\,PB \propto PC\,.\,PD,$$

$$\text{(ii)}\quad PA\,.\,PB \propto PC,$$

where A, B, C, D are fixed points.

[(i) A bicircular quartic such that an infinite number of quadrilaterals can be inscribed in it whose sides pass alternately through the two circular points.

(ii) A bicircular quartic with A and B as singular foci. What is its inverse in respect to A, C or any point ?]

Ex. 28. Find the locus of P, if the product of the tangents from P to two fixed circles (i) is constant, (ii) varies as the square root of the distance of P from a fixed line, (iii) varies as the tangent from P to another fixed circle.

[A bicircular quartic with singular foci at the centres of the two circles.]

Ex. 29. Find the locus of P, if the product of the tangents from P to two fixed circles (i) varies as the distance of P from a fixed line, (ii) varies as the product of the tangents from P to two other fixed circles.

[The reader may consider the question, 'Under what circumstances can a given bicircular quartic be generated by one of the methods of Ex. 27-29?' For instance, any bicircular quartic can be generated by a point P the product of whose distances from the singular foci varies as the tangent from P to a fixed circle, if we allow this circle to be unreal, or allow P to be inside the circle if real; but not always, if we refuse to allow these alternatives.]

Ex. 30. The locus of the intersection of two orthogonal circles one belonging to a given coaxial family and the other belonging to another given coaxial family is a bicircular quartic through the common points and the limiting points of the two families.

[Invert with respect to a common point.]

Ex. 31. Trace the quartic (i) as the intersection of circles.

[It is easy to put the equation in the form $S_1S_2 = S$, where $S_1 = 0$, $S_2 = 0$, $S = 0$ are circles.

Now find the intersections of $tS_1 = S$, $S_2 = t$ for different values of the parameter t.]

§ 4. Quartics with Two Real Nodes.

In our discussions of quartics with two real double points, we shall adopt throughout the following notation. The real double points are A and B. The harmonic conjugate of AB with respect to the tangents at A meets the curve again in M_1 and M_2. The harmonic conjugate of BA with respect to the tangents at B meets the curve again in L_1 and L_2. The lines AM_1M_2, BL_1L_2 meet at C. The tangents at A meet the curve again at E_1 and E_2; the tangents at B meet the curve again at F_1 and F_2. The line E_1E_2 meets the curve again at H_1 and H_2; the line F_1F_2 meets it again at K_1 and K_2. The tangents from A touch at A_1, A_2, A_3, A_4 and the tangents from B touch at B_1, B_2, B_3, B_4. The ranges (AU, M_1M_2) and (BV, L_1L_2) are harmonic. The other diagonals of the quadrilateral formed by the tangents at A and B meet AB at R_1 and R_2 and intersect at C (Fig. 5).

If A and B are not ordinary nodes, a slight modification of the above notation is necessary. Such modification will usually be quite obvious without explanation.

For instance, if B is a flecnode, one of the points F_1 and F_2 will be at B, and the other will be denoted by F. The points K_1 and K_2 do not exist in this case. There are only three tangents from B (other than the tangents at B); their points of contact are B_1, B_2, B_3.

Or again, if B is a cusp, the points F_1 and F_2 coincide at the intersection of the curve with the cuspidal tangent, now called F. The points K_1 and K_2 are the other intersections of the curve with the tangent at F. There are only three tangents from A, whose points of contact are A_1, A_2, A_3. Similarly in other cases.

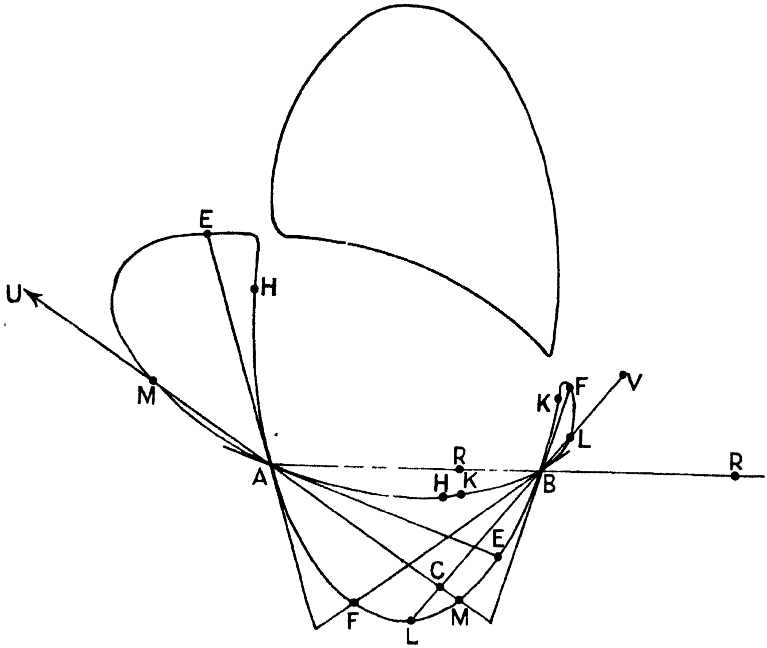


Fig. 5.

Taking ABC as triangle of reference, the coefficients of x^4, y^4, x^3, y^3, x^2yz, xy^2z are zero in the equation of the quartic, which takes the form

$$(x^2+pz^2)(y^2+qz^2)+2m\,xyz^2+2(fx+gy+hz)z^3=0 \quad . \text{ (i)},$$

where $y^2+qz^2=0$ and $x^2+pz^2=0$ are the tangents at A and B.

Replacing x and y by suitable multiples of x and y, we may simplify the equation (i). For instance, we may make f and g

both unity.* But in view of the following sections, it is better to leave f and g general at present.

Writing equation (i) in the form

$$\{xy+(m-t)\,z^2\}^2+z^2\,\{qx^2+py^2+2txy+2fxz+2gyz \\ +(pq+2h-[m-t]^2)\,z^2\}=0,$$

we see that the conic

$$qx^2+py^2+2txy+2fxz+2gyz+(pq+2h-[m-t]^2)\,z^2=0 \quad . \quad \text{(ii)}$$

touches the quartic at its four intersections with

$$xy+(m-t)\,z^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}$$

other than A and B.

The conic (ii) is a line-pair, and therefore represents a pair of bitangents, if

$$t^4-2mt^3+(m^2-2pq-2h)\,t^2+2\,(fg+mpq)\,t \\ +(p^2q^2+2hpq-pqm^2-pf^2-qg^2)=0 \quad . \quad . \quad \text{(iv)}.$$

We shall take the values of t given by (iv) as a, b, c, d.

The pole of AB with respect to the conic (ii), or the intersection of the pair of bitangents, is

$$(gt-pf,\ ft-qg,\ pq-t^2).$$

The locus of this point when t varies is

$$qx^2-py^2+fxz-gyz=0 \quad . \quad . \quad . \quad . \quad . \quad \text{(v)},$$

which passes through the points c, R_1, R_2.

Ex. 1. The points E_1, F_1, E_2, F_2, A, B lie on a conic.
[On $mxy+(fx+gy+hz)\,z=0.$]

Ex. 2. The points of contact of the tangents from C lie on a conic.
[On $qx^2+py^2+2\,(pq+2h)\,z^2+2mxy+3\,(fx+gy)\,z=0.$
Write down the first polar of C.]

Ex. 3. The points A, A_1, A_2, A_3, A_4, L_1, L_2 lie on a conic, and so do B, B_1, B_2, B_3, B_4, M_1, M_2.

The two conics touch the conic of Ex. 1 at A and B respectively.

[On $\qquad py^2+(pq+2h)\,z^2+mxy+fxz+2gyz=0,$
and $\qquad qx^2+(pq+2h)\,z^2+mxy+2fxz+gyz=0.$

Write down the first polar of A and use

$$z^2\,\{py^2+(pq+2h)\,z^2+mxy+fxz+2gyz\}+x\,\{xy^2+qxz^2+myz^2+fz^3\} \\ \equiv (x^2+pz^2)(y^2+qz^2)+2mxyz^2+2\,(fx+gy+hz)\,z^3.]$$

Ex. 4. The three conics of Ex. 2 and 3 pass through the same four points, which also lie on the conic (v).

[Add and subtract the equations of the conics in Ex. 3.]

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* We cannot, however, make $f^2:g^2=q:p$; f or g zero, &c.

Ex. 5. The points $A_1, A_2, A_3, A_4, B, H_1, H_2$ lie on a conic which touches AH_1, AH_2, BC.

[On $(f^2+qm^2)x^2+2(fg-hm)xy+2(qmg+hf)xz$
$$-(pf^2+pqm^2+2hm^2-2fgm)z^2=0.$$
The conic $B_1\,B_2\,B_3\,B_4\,A\,K_1\,K_2$ is
$(g^2+pm^2)y^2+2(fg-hm)xy+2(pmf+hg)yz$
$$-(qg^2+pqm^2+2hm^2-2fgm)z^2=0.$$

If we write $X\equiv(f^2+qm^2)x+(fg-hm)y+(qmg+hf)z$, the quartic becomes $(y^2+qz^2)X^2+2u_3X+(y^2+qz^2)v_2=0$, where
$v_2\equiv\{(fg-hm)y+(qmg+hf)z\}^2+(f^2+qm^2)(pf^2+pqm^2+2hm^2-2fgm)z^2$
and u_3 is homogeneous of degree 3 in y and z.

Hence the required conic is $X^2=v_2$ by § 1; the line E_1E_2 being $X=0$, and the lines AH_1, AH_2 being $v_2=0$.]

Ex. 6. A conic touches the quartic at L_1, L_2, M_1, M_2.

[Put $t=m$ in (ii).]

Ex. 7. The conics (ii) meet the sides of the triangle ABC in involutions of which BV, AU, R_1R_2 are double points.

Ex. 8. Through the intersections of any two of the conics (ii) passes a conic touching CA and CB at A and B.

[$\{qx^2+py^2+2t_1xy+2fxz+2gyz+(pq+2h-[m-t_1]^2)z^2\}$
$$-\{qx^2+py^2+2t_2xy+2fxz+2gyz+(pq+2h-[m-t_2]^2)z^2\}$$
$$\equiv(t_1-t_2)\{2(xy+[m-t]z^2)+(2t-t_1-t_2)z^2\}.]$$

Ex. 9. If P, Q are any two points dividing R_1R_2 harmonically, a conic can be drawn through P, Q and the intersections of any one of the conics (ii) with any one of the conics (iii). It touches one of the conics (ii) at P and Q.

[See Ex. 8.]

Ex. 10. Through the points P, Q of Ex. 9 a conic can be drawn passing through the four points of contact of any one of the conics (ii) with the quartic.

[In Ex. 8 put $t_1=t$.]

Ex. 11. A conic can be drawn through the points of contact of any two of the conics with the quartic.

[If the conics are given by $t=t'$ and $t=t''$, put $t=t_1=t'$ and $2t_2=t'+t''$ in Ex. 8.]

Ex. 12. The conic through C and the intersections of any two of the conics (ii) has a fixed tangent at C and divides AU, BV harmonically.

Ex. 13. The points of contact of any pair of bitangents lie on a conic touching CA at A and CB at B.

[On conic (iii), where $t=a, b, c$, or d.]

Ex. 14. The intersections of the four pairs of bitangents with AB form an involution whose double points are R_1 and R_2.

[See Ex. 7.]

Ex. 15. The four intersections of the two pairs of bitangents lie on a conic touching CA at A and CB at B.

[See Ex. 8.]

Ex. 16. The conics through C and the intersections of any two pairs of bitangents touch one another at C and divide AU, BV harmonically.

[See Ex. 12.]

Ex. 17. The points of contact of any two pairs of bitangents lie on a conic.

[See Ex. 11.]

Ex. 18. The conic $CE_1E_2F_1F_2$ touches the conics of Ex. 12 at C. The tangent to this conic and the tangent to conic (v) at C divide AB harmonically.

[The conic is

$$h(f^2+qm^2)x^2+h(g^2+pm^2)y^2+(2fgh-mh^2-pqm^3)xy$$
$$+(gh-pqm^2)(fx+gy)z = 0.]$$

Ex. 19. Show that the pencils of tangents from A and B have the same cross-ratio.

[The tangents are

$$py^4+2gy^3z+(2h+2pq-m^2)y^2z^2+2(gq-mf)yz^3+(2hq+pq^2-f^2)z^4=0,$$
$$qx^4+2fx^3z+(2h+2pq-m^2)x^2z^2+2(fp-mg)xz^3+(2hp+p^2q-g^2)z^4=0.$$

Now use Ch. I, § 11.

Another proof consists in projecting A and B into the circular points and using the fact that the foci of a bicircular quartic lie by fours on circles.]

Ex. 20. The left-hand side of (ii) may be written in the form $uv+kz^2$, where kpq is the left-hand side of (iv) and

$$u \equiv \sqrt{q\tan\tfrac{1}{2}\alpha}\,(x-\frac{\tan\alpha}{\sqrt{qp}}gz)+\sqrt{p\cot\frac{\alpha}{2}}\,(y+\frac{\tan\alpha}{\sqrt{pq}}fz),$$
$$v \equiv \sqrt{q}\cot\tfrac{1}{2}\alpha\,(x+\frac{\tan\alpha}{\sqrt{pq}}gz)+\sqrt{p\tan\frac{\alpha}{2}}\,(y-\frac{\tan\alpha}{\sqrt{pq}}fz),$$
$$t = \sqrt{pq}\,\operatorname{cosec}\alpha.$$

[Taking $t = a, b, c, d$, we get the eight bitangents.]

Ex. 21. The quartic is the envelope of the conic

$$\rho^2uz+2\rho\{xy+(m-t)z^2\}+vz=0;$$

where t is any root of (iv), and ρ is a parameter.

The locus of the pole of $z=0$ with respect to this conic is a conic touching the tangents at A and B to the quartic.

[$qx^2+py^2-2txy+(pq-t^2)z^2=0.$]

Ex. 22. The sixteen intersections of tangents to the quartic from A and B lie by fours on four conics touching the tangents at A and B.

[The intersections are the centres of the line-pair conics of the family of Ex. 21, and these lie on the pole-locus.]

Ex. 23. The points A, A_1, A_2, A_3, A_4, B, B_1. B_2, B_3, B_4, U, V lie on a cubic. It passes through the intersection of AB with the tangent at C to the conic at Ex. 12 and the intersections of CA and CB with the tangents at B and A to the conic of Ex 1.

[Dividing

$$(xy^2+qxz^2+myz^2+fz^3)(yx^2+pyz^2+mxz^2+gz^3)$$
$$-xy\{(x^2+pz^2)(y^2+qz^2)+2mxyz^2+2(fx+gy+hz)z^3\}$$

by z^3 we obtain the cubic

$$xy(fx+gy)=z\{qmx^2+(m^2-2h)xy+pmy^2\}$$
$$+z^2\{(qg+mf)x+(pf+mg)y\}+fgz^3.]$$

Ex. 24. If one polygon of $2n$ sides can be inscribed in a binodal quartic so that the sides go alternately through the nodes, an infinite number of such polygons can be inscribed.

[Project the Steiner points of a polygon inscribed in a cubic (Ch. XVI, § 6, Ex. 1) into the circular points and invert.]

Ex. 25. All quartics with two given nodes and passing through seven other fixed points, pass through an eighth fixed point.

[Project the nodes into the circular points and invert with respect to one of the fixed points.]

Ex. 26. Discuss the quartic for which $f^2/g^2 = q/p$.

[The quartic has either two crunodes or two acnodes. In the first case we may take $p = q = -1$, $g = f$. One root of (iv) is $t = -1$, the corresponding pair of bitangents meeting at $(1, -1, 0)$. The locus of the pole of AB with respect to the conics (ii) is $x = y$. The quartic can be projected so as to be symmetrical. Similarly in the second case.]

Ex. 27. Discuss the quartic for which $fg = hm$.

[The conics of Ex. 1 and 5 degenerate.]

§ 5. Quartics with a Node and a Cusp.

In § 4 A and B were any two real double points of a quartic. If A is a node and B is a cusp, we have $p = 0$ and we may also take $f = 1$, $g = 1$, as explained in § 4.

The equation (i) of the quartic becomes

$$x^2(y^2+qz^2)+2m\,xyz^2+2(x+y+hz)z^3 = 0 \quad . \quad . \quad \text{(i)};$$

while equations (ii), (iii), and (iv) become

$$qx^2+2txy+2xz+2yz+(2h-[m-t]^2)z^2 = 0 \quad . \quad \text{(ii)},$$

$$xy+(m-t)z^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)},$$

$$t^4-2mt^3+(m^2-2h)t^2+2t-q = 0 \quad . \quad . \quad . \quad \text{(iv)}.$$

If t is any one of the roots a, b, c, d of (iv), (ii) represents a line-pair which must evidently be

$$(z+tx)\{qtx+2t^2y+(2t-q)z\} = 0,$$

and by § 4 this touches the quartic where it meets (iii).

Taking $t = a$, we obtain a tangent from the cusp $z+ax = 0$ touching at the point $B_1\ (-1, ma^2-a^3, a)$, and a corresponding bitangent b_1 with equation

$$qax+2a^2y+(2a-q)z = 0$$

touching where

$$2a^2y^2+(2a-q)yz+qa(a-m)z^2 = 0.$$

Similarly taking $t = b$ we get the tangent $z+bx = 0$ from the cusp touching at B_2 and the corresponding bitangent b_2.

$$qbx+2b^2y+(2b-q)z = 0;$$

and so for $t = c$, $t = d$.

The line $2y = \epsilon z$ meets the curve (i) again in coincident points, if

$$\epsilon^3+(2h-m^2)\epsilon^2+4(q-m)\epsilon+4(2qh-1) = 0.$$

This is readily proved to be an equation in ϵ whose roots are

$$ad+bc, \quad bd+ca, \quad cd+ab.$$

Hence the tangents from the node A are

$$2y = (ad+bc)\,z, \text{ \&c.}$$

The points of contact A_1, A_2, A_3 will be found to be

$$(2b+2c-2a-2d,\ 2ad-2bc,\ a^2d^2-b^2c^2), \text{ \&c.}$$

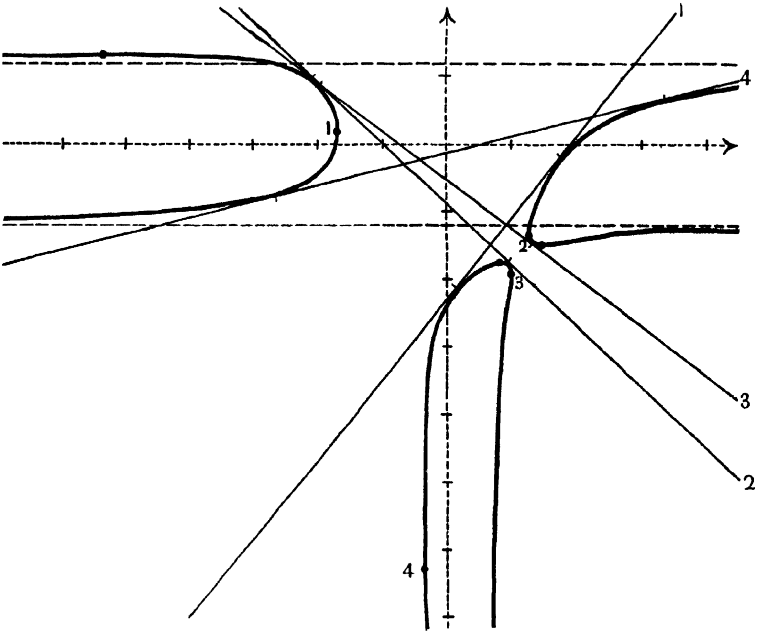

Fig. 6.
$100x^2y^2+2y(100+90x)-144x^2+36x+469=0.$

In Fig. 6 is shown the quartic

$$100x^2y^2+2y(100+90x)-144x^2+36x+469=0.$$

The tangents from the cusp $(0, \infty)$ are

$$3x=-5, \quad 4x=5, \quad x=1, \quad 3x=-1$$

and the corresponding bitangents b_1, b_2, b_3, b_4 are

$$12x-10y=23, \quad 9x+10y=-9, \quad 36x+50y=-27,$$
$$12x-50y=7.$$

The tangents from and at the node $(\infty, 0)$ are respectively

$$10y = 13, \quad 2y = -3, \quad 50y = -87, \quad \text{and} \quad 5y = \pm 6.$$

In Fig. 6 B_1, b_1, &c., are represented by their suffixes only.

Ex. 1. Apply the method of § 2 to obtain the bitangents of the quartic (1) of § 5 and the tangents from A.

[$yx^2 + z^3 + mz^2x = 0$ is the polar cubic of B and the quartic is

$$(yx^2 + z^3 + mz^2x)^2 = (z + ax)(z + bx)(z + cx)(z + dx).$$

Hence $2(yx^2 + z^3 + mz^2x) = z^2(z + ax) + (z + bx)(z + cx)(z + dx)$ is a bitangent touching where

$$z^2(z + ax) = (z + bx)(z + cx)(z + dx).$$

Similarly $2(yx^2 + z^3 + mz^2x) = z(z + ax)(z + dx) + z(z + bx)(z + cx)$ is a tangent from A touching where

$$(z + ax)(z + dx) = (z + bx)(z + cx).$$

The reader may consult *Messenger Math.*, xlvi (1916), p. 81.]

Ex. 2. The conic $A_1 A_2 A_3 H_1 H_2$ touches the quartic at B and touches AH_1, AH_2.

[See § 4, Ex. 5.]

Ex. 3. The conic $ABE_1 E_2 F$ touches the quartic at F.

[See § 4, Ex. 1.]

Ex. 4. The points of contact of the tangents from C lie on a conic through B.

[See § 4, Ex. 2.]

Ex. 5. The pencils $B(B_1 B_2 B_3 B_4)$ and $A(A_1 A_2 A_3 B)$ have the same cross-ratio.

Ex. 6. The conic through B, A, B_1 and the points of contact of the bitangent b_1 touches AC at A and BC at B.

[$xy + (m - a)z^2 = 0$.]

Ex. 7. B, B_1, B_2 and the points of contact of b_1, b_2 lie on a conic.

[$qx^2 + (2h - m^2 + am + bm - ab)z^2 + 2yz + (a + b)xy + 2xz = 0$.]

Ex. 8. The cross-ratio of the pencil formed by the tangents from B is equal to the cross-ratio of the range formed by the intersections of AB with the bitangents.

Ex. 9. The conic touching AB, b_1, b_2, b_3, b_4 touches AB at B, and ouches AC.

[$2\lambda^2 + q\mu\nu = 2\lambda\mu$.]

Ex. 10. The diagonals of the quadrilateral formed by the bitangents pass respectively through A_1, A_2, A_3.

Ex. 11. Four, two, or none of the bitangents are real according as our, two, or none of the tangents from the cusp are real.

Ex. 12. The points of contact of the two real conics through A, B, F sculating the curve (not at A, B, or F) lie on a line through A.

[In Ch. IX, § 1, take ABF as the triangle ABC.]

§ 6. Cartesian Curves.

If a quartic has a cusp at each circular point ω, ω', its inverse with respect to any point Q has Q as a focus (Ch. V, § 4) and is a bicircular quartic (§ 3, Ex. 1), which is self-inverse for a real circle j through Q. The quartic with ω and ω' as cusps is therefore symmetrical about the line which is the inverse of j.

Taking the line as axis of x, the equation of the quartic takes the form

$$(x^2+y^2)^2+p(x^2+y^2)x+ax^2+by^2+2gx+c=0 \quad . \quad \text{(i)}.$$

The curve is of class 6, and therefore there are three tangents from ω other than the tangent at ω. Hence the curve has one distinct singular focus O, the intersection of the tangents at ω and ω', and three real ordinary foci.

Considerations of symmetry show that there are only two cases to consider.

(1) The singular focus O and the real ordinary foci A, B, C lie on the axis of x.

(2) The singular focus O and one real ordinary focus C lie on the axis of x, which is the perpendicular bisector of the line joining the real ordinary foci E, F.

Let us consider case (1).

When we invert with respect to Q, the real foci are Q and the inverses of A, B, C. These lie on the real circle j. Hence the quartic consists of *two* ovals (§ 3).

Let us take O as origin and A, B, C as the points $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$. Then $y = \pm ix$ are asymptotes twice over, which gives $p = 0$ and $a = b$ in (i). Also $y = i(x-d)$ touches the curve (i), if

$$4gd^3+(4c-a^2)d^2-2agd-g^2=0.$$

This equation in d has the roots α, β, γ; from which it readily follows that (i) now takes the form

$$(x^2+y^2-\beta\gamma-\gamma\alpha-\alpha\beta)^2+4\alpha\beta\gamma(2x-\alpha-\beta-\gamma)=0 \quad . \quad \text{(ii)}.$$

The curve is called a 'Cartesian curve'.*

We now show that:

The locus of a point P whose distances from two fixed points A, B are connected by a linear relation is a Cartesian curve.

Suppose the relation is

$$\pm m \,.\, PA \pm l \,.\, PB = n \,.\, AB \quad . \quad . \quad . \quad . \quad \text{(iii)}.$$

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* Or 'Cartesian oval'; but it consists of *two* ovals.

Take a point O on AB such that

$$OA : OB = l^2 : m^2,$$

and let C be a point on AB such that

$$l^2 : m^2 : n^2 = OA : OB : OC.$$

Then the equation of the locus P, when O is taken as origin and A, B, C as $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$ is

$$\pm \beta^{\frac{1}{2}}\{(x-\alpha)^2+y^2\}^{\frac{1}{2}} \pm \alpha^{\frac{1}{2}}\{(x-\beta)^2+y^2\}^{\frac{1}{2}} = \gamma^{\frac{1}{2}}(\beta-\alpha),$$

which when rationalized proves to be the same as (ii).

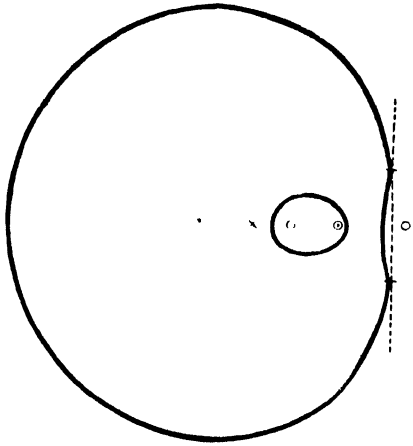

Fig. 7.
$(x^2+y^2-61)^2+144(4x-29)=0.$
Singular focus (0, 0); ordinary foci (2, 0), $(\frac{9}{2}, 0)$, (8, 0).

Conversely, the distances of any point P on a Cartesian curve from the foci A, B are connected by the linear relation

$$\pm OB^{\frac{1}{2}} . PA \pm OA^{\frac{1}{2}} . PB = OC^{\frac{1}{2}} . BA \quad . \quad . \quad . \quad \text{(iv)},$$

where O is the singular focus, and C the third real ordinary focus.

A quartic with two cusps has one bitangent. The bitangent of (ii) is evidently

$$2x = \alpha + \beta + \gamma.$$

In Fig. 7 O is (0, 0), A is (2, 0), B is $(\frac{9}{2}, 0)$, C is (8, 0). The curve meets $OABC$ at points whose distances from O are -13, 1, 5, 7. We have $\pm 3PA \pm 2PB = 10$, &c.

Case (2), in which the three real foci of (i) are not collinear, is not so interesting geometrically. The real foci E, F, C may be taken as $(\xi, \pm\eta)$, $(0, \gamma)$, and the singular focus O as origin. Then $(\xi \pm i\eta, 0)$ are also foci. Hence the equation of the curve is obtained by replacing α, β in (ii) by $\xi \pm i\eta$. It is

$$(x^2+y^2-2\xi\gamma-\xi^2-\eta^2)^2+4\gamma(\xi^2+\eta^2)(2x-2\xi-\gamma) = 0 \quad . \quad \text{(v)},$$

the bitangent being

$$2x = 2\xi+\gamma.$$

The curve consists of a single oval. (See *Proc. London Math. Soc.*, iii, p. 115, for diagrams of this case.)

Ex. 1. What does the locus of P become, if in (iii) l, m, or n is zero ?

Ex. 2. The distances from a given focus of the intersections of a Cartesian curve with a variable line have a constant sum.

[Use polar coordinates, taking the focus as pole.]

Ex. 3. The points of contact of (ii) with its bitangent are real if

$$2(\beta\gamma+\gamma\alpha+\alpha\beta) > (\alpha^2+\beta^2+\gamma^2).$$

Ex. 4. Show that, if

$$PA+PB = 2u, \qquad PB-PA = 2v, \qquad AB = 2c,$$

where A and B are fixed points and P a variable point, then the orthogonal trajectory of the family of curves with differential equation

$$M\,du+N\,dv = 0 \quad \text{is} \quad N(v^2-c^2)\,du+M(u^2-c^2)\,dv = 0.$$

Ex. 5. Find the orthogonal trajectories of Cartesian curves (i) with given singular focus and two given ordinary foci, (ii) with two given ordinary foci and passing through a fixed point.

[Use (iv) to obtain M and N in Ex. 4.]

Ex. 6. Two Cartesian curves with the same real ordinary foci cut orthogonally.

[Invert with respect to an intersection and use Ch. XIV, § 3, Ex. 10. Inverting with respect to any point, we see that any two bicircular quartics with the same four real concyclic foci cut orthogonally.]

Ex. 7. What relation connecting the distances of a point on the curve from two real foci replaces (iv) in the case of the quartic (v) ?

[Use Ch. XIV, § 3, Ex. 25.]

Ex. 8. By inversion of (iv) obtain the results of § 3, Ex. 17.

Ex. 9. Discuss the modifications which must be made in § 3 when the bicircular quartic is a Cartesian curve.

[§ 3 (ii) becomes any circle with centre on $y = 0$ touching the curve at two points only. The roots of § 3 (iv) are

$$\beta\gamma+\gamma\alpha+\alpha\beta, \qquad -\beta\gamma+\gamma\alpha+\alpha\beta, \qquad \beta\gamma-\gamma\alpha+\alpha\beta, \qquad \beta\gamma+\gamma\alpha-\alpha\beta.$$

The first root makes (ii) become the bitangent, the other three make (ii) become the circular lines through a focus.

§ 3 (v) becomes the circular lines through O or else § 3 (v), (vi) become the circles

$$x^2+y^2 = \beta\gamma, \qquad (x-\alpha)^2+y^2 = (\alpha-\beta)(\alpha-\gamma), \qquad \&\text{c}.$$

Hence the Cartesian curve is in three different ways the envelope of a circle whose centre lies on a fixed circle with centre at the singular focus, and which cuts orthogonally a fixed circle with its centre at a focus. The other two foci are the limiting points of the fixed circles.]

Ex. 10. Show that the locus of P is a quartic with cusps at ω and ω', if

(i) The tangents from P to two fixed circles are connected by a linear relation.

(ii) The square of the tangent from P to one circle varies as the tangent from P to another circle.

(iii) The fourth power of the tangent from P to a circle varies as the distance of P from a line.

Ex. 11. Find the locus of a point whose normal distances from two fixed circles have a constant ratio.

Ex. 12. A man is in a pond at A, swims to the bank at P, and runs to a point B on the land. If his time from A to B is the same for all positions of P, find the shape of the pond.

Ex. 13. Find the surface separating two homogeneous isotropic media, if rays of light emanating from a point in one medium are brought accurately to a focus in the other.

Ex. 14. Given one focus and three points of a Cartesian curve, find the locus of the other foci.

§ 7. Quartics with Two Real Cusps.

Suppose that in § 4 both A and B were cusps. Then C is the intersection of the tangents at A and B. We have now $p = 0$, $q = 0$. As explained in § 4, we may suppose $f = 1$, $g = 1$; and we shall put

$$k = h - \tfrac{1}{2}m^2.$$

The equation (i) of the quartic becomes

$$(xy + mz^2)^2 + 2(x + y + kz)z^3 = 0 \quad . \quad . \quad . \quad . \quad \text{(i)};$$

while equations (ii), (iii), (iv), and (v) become

$$txy + xz + yz + (k + mt - \tfrac{1}{2}t^2)z^2 = 0 \quad . \quad . \quad . \quad \text{(ii)},$$

$$xy + (m - t)z^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)},$$

$$t\{t^3 - 2mt^2 - 2kt + 2\} = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)},$$

$$z(x - y) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}.$$

If we take the root $t = 0$ of (iv), (ii) becomes $z = 0$ and the bitangent

$$x + y + kz = 0$$

of the quartic (i). In fact from equation (i) it is obvious that this line is the bitangent of the quartic, the points of contact W_1 and W_2 lying on

$$xy + mz^2 = 0.$$

If we take the other roots $t = a, b, c$ of (iv), (ii) gives the tangents

$$z + ay = 0, \quad z + by = 0, \quad z + cy = 0$$

from A touching at A_1, A_2, A_3 and the tangents

$$z + ax = 0, \quad z + bx = 0, \quad z + cx = 0$$

from B touching at B_1, B_2, B_3.

The coordinates of A_1 are $(a^3-ma^2, 1, -a)$, and so for the other points of contact.

The reader may illustrate the properties of the curve by tracing

$$9x^2y^2+96xy+144(x+y)+496 = 0.$$

The bitangent is

$$3x+3y+5 = 0;$$

while the tangents from the cusp $(0, \infty)$ are $x = 2$, $x = -\frac{2}{3}$, $x = -3$; and so for the cusp $(\infty, 0)$.

Ex. 1. The points A, B, E, F, W_1, W_2 lie on a conic.

[Putting $z = 1$ we see that the quartic can be projected so as to have an axis of symmetry.

The reader can write down a large number of theorems which follow from this fact. For instance, A_1, A_2, A_3, B_1, B_2, B_3 lie on a conic, &c., &c.]

Ex. 2. A, B, A_1, A_2, A_3, L lie on a conic.

[See § 4, Ex. 3.]

Ex. 3. The conic W_1, W_2, A_1, A_2, A_3 touches the quartic at A.

[It is $xy+mz^2 = 2y(x+y+kz)$.]

Ex. 4. The points of contact of the other four tangents which can be drawn to the curve from any point of the bitangent lie on a conic through A and B.

[The polar cubic of the point (x', y', z') on $x+y+kz = 0$ with respect to (i) meets (i) on the conic

$$2z(y'x+x'y+2mz'z) = 3z'(xy+mz^2).]$$

§ 8. Cassinian Curves.

Consider a quartic with a biflecnode at each circular point ω, ω'. It has two real singular foci F_1, F_2. We may choose axes of reference such that these singular foci are $(\pm c, 0)$. Writing down the general equation of a bicircular quartic, and expressing the fact that the four lines

$$y = \pm i(x \pm c)$$

meet the curve only at ω or ω', we see readily (cf. § 3 (i)) that the quartic has the equation

$$(x^2+y^2)^2-2c^2(x^2-y^2)+c^4 = a^4 \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

The left-hand side of (i) is at once proved to be the square of the product of the distances of the point (x, y) from $(\pm c, 0)$. Hence, if P is any point on (i),

$$PF_1 . PF_2 = a^2.$$

The curve is called a 'Cassinian Curve'.* If $c = a$, the curve is a lemniscate of Bernouilli.

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* Or 'Cassinian Oval'. But it consists of *two* ovals if $c > a$.

The shape of the curve is shown on Fig. 8 for the cases $a/c = \sqrt{\frac{1}{2}}, \sqrt{\frac{7}{8}}, 1, \sqrt{\frac{9}{8}}, \sqrt{\frac{3}{2}}, \sqrt{2}, \sqrt{\frac{5}{2}}, \sqrt{3}$.

The most interesting properties of the Cassinian curve are given in the following examples.*

In order to keep this chapter within reasonable compass, we shall confine ourselves to a very brief account of the other varieties of quartic with unit deficiency. The reader who is interested will find in the examples their more important properties.

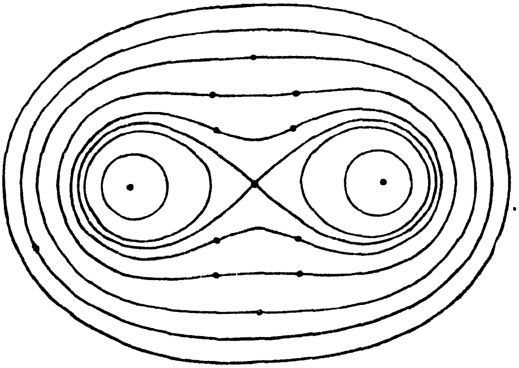

Fig. 8.

Ex. 1. The real ordinary foci S_1, S_2 of (i) are $(\pm d, 0)$ where

$$c^2 d^2 = c^4 - a^4,$$

if $c > a$; and are $(0, \pm d)$ where $c^2 d^2 = a^4 - c^4$, if $c < a$.

Ex. 2. If P is any point of (i) and O is the centre of (i), i.e. the middle point of $F_1 F_2$ and of $S_1 S_2$, then

$$c^2 . PS_1 . PS_2 = a^2 . PO^2.$$

Ex. 3. The angle between OP and the normal at P is the difference of the angles OPF_1 and OPF_2.

Ex. 4. A family of Cassinian curves has given singular foci F_1, F_2. Show that

(i) Their orthogonal trajectories are rectangular hyperbolas with $F_1 F_2$ as diameter.

(ii) The locus of the inflexions is a lemniscate of Bernouilli.

(iii) The locus of the points of contact of tangents parallel to $F_1 F_2$ is the circle on $F_1 F_2$ as diameter and the perpendicular bisector of $F_1 F_2$.

[The polar and pedal equations of (i) when O is pole are

$$2r^2c^2 \cos 2\theta = r^4 + c^4 - a^4 \quad \text{and} \quad 2a^2pr = r^4 + a^4 - c^4,$$

giving $(3r^4 + c^4 - a^4)p = 2a^2r^3$. The inflexions are shown in Fig. 8; their locus is $r^2 + c^2 \cos 2\theta = 0$.]

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* The reader may extend these results to the 'n-poled Cassinoid', i.e. the locus of a point whose distances from the vertices of a regular polygon have a constant product. See *Messenger Math.*, xlviii (1919), p. 184.

Ex. 5. Find the orthogonal trajectories and the locus of the inflexions of Cassinian curves with given ordinary foci.

[The curves are the inverses of the family in Fig. 8 with respect to O.]

Ex. 6. If $ABCD$ is a parallelogram, find the locus of P when

(i) $PA.PC : PB.PD$ is constant.

(ii) The sum of the angles made by PA and PC with a fixed line less the sum of the angles made by PB and PD with the line is constant.

(iii) The Cassinian curve through P with singular foci A, C and the Cassinian curve through P with singular foci B, D cut at a given angle.

[Each locus is a Cassinian curve.]

Ex. 7. Find the locus of P, if the polar conic of P with respect to a given Cassinian curve has a constant eccentricity.

[A Cassinian curve with the singular foci of the given curve as ordinary foci.]

Ex. 8. The locus of the foci of a variable conic concentric with and having double contact with two given confocal conics is a Cassinian curve with singular foci at the foci of the given confocals.

[If (x, y) is a focus of $A\lambda^2+2H\lambda\mu+B\mu^2 = 1$ which has double contact with the confocals $\alpha\lambda^2+\beta\mu^2 = 1$, $(\alpha+k)\lambda^2+(\beta+k)\mu^2 = 1$, we have $x^2-y^2 = A-B$, $xy = H$, $(\alpha-A)(\beta-B) = (\alpha+k-A)(\beta+k-B) = H^2$. Now eliminate A, B, H.]

Ex. 9. The locus of the intersections of two circles having their centres at the foci of a given conic and touching any tangent of the conic is a Cassinian curve.

Ex. 10. The locus of the foci of an ellipse with a given director circle and passing through a given point is a Cassinian curve.

[If C is the centre and r the radius of the director circle, P the fixed point and C bisects PQ, while S and S' are the foci of the ellipse,

$$PS\,.\,QS = PS\,.\,PS' = r^2-CP^2.]$$

Ex. 11. What do the results of § 3 become in the case of a Cassinian curve?

[§ 3 (iv) gives $t = \pm c^2$, or $\pm(c^4-a^4)^{\frac{1}{2}}$.
The Cassinian curve is the envelope of a circle which cuts orthogonally the director circle of a hyperbola and whose centre lies on the curve.]

§ 9. Quartics with Two Real Biflecnodes.

Ex. 1. A quartic has real biflecnodes A and B. With the notation of § 4 its equation is $(x^2+pz^2)(y^2+qz^2)+2hz^4 = 0$.

Ex. 2. Any line through A is divided harmonically by BC and the curve.

Ex. 3. The intersections of the curve with any line through C form two pairs of an involution with one double point at C and the other on AB.

Ex. 4. The tangents from A to the quartic are AL_1 and AL_2. The tangents from B are BM_1 and BM_2.

Ex. 5. A conic touches the quartic at L_1, L_2, M_1, M_2 having ABC as self-conjugate triangle.

[$qx^2+py^2+(pq+2h)z^2 = 0$.
Aliter, project the quartic into the curve with double symmetry obtained by putting 1 for z; and so for other examples.]

Ex. 6. The eight inflexions other than those at A and B lie on a conic with ABC as self-conjugate triangle.

$[3qx^2+3py^2+2(pq+2h)z^2=0.]$

Ex. 7. Four of the bitangents of the quartic pass through C. Their eight points of contact lie on a conic having ABC as self-conjugate triangle.

[In § 4 we have $m=f=g=0$. Two roots of § 4 (iv) are

$$t=\pm(pq+2h)^{\frac{1}{2}},$$

giving the bitangents

$$qx^2\pm 2(pq+2h)^{\frac{1}{2}}xy+py^2=0,$$

which touch the curve where

$$qx^2+py^2+2(pq+2h)z^2=0.]$$

Ex. 8. The other four bitangents form a quadrilateral of which two vertices lie on CA, two on CB, and two on AB. The two latter are on the diagonals of the quadrilateral formed by the tangents at A and B. The points of contact of the bitangents lie on a conic having ABC as self-conjugate triangle.

If $9h=8pq$ each of these bitangents meets one bitangent of Ex. 7 on a tangent at A and another on a tangent at B.

[Two roots of § 4 (iv) are $t=\pm(pq)^{\frac{1}{2}}$, giving the bitangents

$$qx^2\pm 2(pq)^{\frac{1}{2}}xy+py^2+2hz^2=0$$

which touch the curve where $qx^2+py^2+2(pq+h)z^2=0$.

Illustrate by tracing $x^2y^2-9x^2-y^2+25=0$ with bitangents $x=y$, $9x=y$, $3x+y=4$, &c.]

Ex. 9. The conics of Ex. 5, 6, 7, 8 touch at two points on AB.

$[qx^2+py^2=z=0.]$

Ex. 10. Discuss the conditions under which the inflexions, bitangents, and points of contact of the bitangents are real.

Ex. 11. A quartic cannot have a biflecnode and a flecnode.

§ 10. Quartics with Two Unreal Flecnodes.

Ex. 1. A quartic has flecnodes at the circular points If the intersection O of the inflexional tangents at the flecnodes is taken as origin and the intersection S of the non-inflexional tangents as $(-g, 0)$, the equation of the curve takes the form

$$(x^2+y^2)(x^2+y^2+2gx+c)=k.$$

Ex. 2. The quartic has three real ordinary foci A, B, C. If these are $(\alpha, 0)$, $(\beta, 0)$, $(\gamma, 0)$, the quartic is

$$(x^2+y^2)\{(\alpha^2+\beta^2+\gamma^2-2\beta\gamma-2\gamma\alpha-2\alpha\beta)(x^2+y^2)+8\alpha\beta\gamma x$$
$$-2\alpha\beta\gamma(\alpha+\beta+\gamma)\}+\alpha^2\beta^2\gamma^2=0.$$

The directrices are

$$2(\beta+\gamma-\alpha)x=\beta\gamma+\gamma\alpha+\alpha\beta-\alpha^2, \text{ \&c.}$$

Ex. 3. If P is any point on the quartic of Ex. 2

$$\frac{BC\,.\,AP}{\sqrt{OA}}\pm\frac{CA\,.\,BP}{\sqrt{OB}}\pm\frac{AB\,.\,CP}{\sqrt{OC}}=0.$$

[Rationalizing $(\beta-\gamma)\alpha^{-\frac{1}{2}}(x^2+y^2-2\alpha x+\alpha^2)^{\frac{1}{2}}+\dots+\dots=0$, we get the equation of Ex. 2.]

Ex. 4. Discuss the case in which the real foci are $(\xi, \pm\eta)$, $(\gamma, 0)$. [Cf. Ch. XIV, § 3, Ex. 25.]

Ex. 5. The quartic of Ex. 1 can be generated as the locus of P in one of three ways.

(1) The product of PO and the tangent from P to a fixed circle with centre S is constant.

(2) $PO \,.\, PS \propto$ the tangent from P to a fixed circle with centre O.

(3) $PO \,.\, PS \propto$ the chord through P perpendicular to PS of a fixed circle with centre S.

[The three cases are given by

(1) $k > 0$, $g^2 > c$. (2) $k < 0$, $g^2 > c$. (3) $k > 0$, $g^2 < c$.

Diagrams will be found in *Proc. London Math. Soc.*, xii, p 22.]

§ 11. Quartics with Two Real Flecnodes.

Ex. 1. A quartic has two real flecnodes A and B. With the notation of § 4 its equation may be written

$$(x^2 - z^2)(y^2 - z^2) + 2mz^2(xz + yz + xy) + 2hz^4 = 0.$$

[For the following examples see § 4 or use the symmetry in x and y. The non-inflexional tangents at A and B meet the curve again in E and F, and the inflexional tangents meet at T.]

Ex. 2. $A, A_1, A_2, A_3, L_1, L_2$ lie on a conic touching AT at A.

Ex. 3. A, B, E, F lie on a conic touching AT, BT.

Ex. 4. The conic $AA_1A_2A_3B$ touches AE, BC.

Ex. 5. $A_1, A_2, A_3, B_1, B_2, B_3$ lie on a conic.

Ex. 6. E, F, L_1, L_2, M_1, M_2 lie on a conic.

Ex. 7. The tangents from C form an involution and their points of contact lie on a conic.

Ex. 8. The tangents from the intersection of AE and BF form an involution.

Ex. 9. Three pairs of bitangents intersect on CT and the other pair on AB.

[Putting $p = q = -1$, $f = g = m$ in § 4 (iv), one value of t is -1.]

§ 12. Quartics with a Node and a Biflecnode.

Ex. 1. A quartic has a node A and a biflecnode B. With the notation of § 4 its equation may be written

$$(x^2 + pz^2)(y^2 + qz^2) + 2(x + hz)z^3 = 0.$$

[The quartic may be projected into the symmetrical curve obtained by putting $z = 1$.]

Ex. 2. Any line through B is divided harmonically by the curve and AC.

Ex. 3. The two tangents from B touch at the points M_1, M_2.

Ex. 4. The four tangents from A form an involution pencil whose double rays are AB, AC. The tangents at A belong to the involution. The points of contact lie on a pair of lines through B and on a conic touching AB at A.

Ex. 5. B, E_1, E_2 are collinear.

Ex. 6. The intersections of the bitangents lie on AC or by pairs on lines through B.

[Putting $m = y = 0,\ f = 1$ in § 4 (iv), the bitangents are

$$qx^2 + py^2 + 2txy + 2xz + (pq + 2h - t^2) z^2 = 0,$$

where $t^2 = pq + h \pm (h^2 + p)^{\frac{1}{2}}$.]

§ 13. Quartics with a Flecnode and another Double Point.

Ex. 1. The quartic has a node A and a flecnode B. With the notation of § 4 its equation may be written

$$(x^2 - z^2)(y^2 + qz^2) + 2xyz^2 + 2(fx + y + hz) z^3 = 0.$$

Ex. 2. The conic ABE_1E_2F touches the inflexional tangent at B.

Ex. 3. The points $B, B_1, B_2, B_3, M_1, M_2$ lie on a conic touching the inflexional tangent at B.

Ex. 4. The conic $BB_1B_2B_3A$ touches AC at A and the non-inflexional tangent at B.

Ex. 5. Transform the curve of Ex. 1 into a cubic by quadratic transformation, and investigate properties of the curve by this means.

[Replace x by $z(z - x)/x$.]

Ex. 6. A quartic cannot have a cusp and a biflecnode.

Ex. 7. The quartic of § 5 has a cusp and a flecnode, if

$$(ad - bc)(bd - ca)(cd - ab) = 0.$$

The tangents from B form an involution with BA, BC. The bitangents meet AB in two pairs of points forming an involution with A, B.

§ 14. Quartics with a Tacnode.

We now consider quartics with a tacnode, i.e. a double point at which two linear branches touch. We take the tacnode as $C(0, 0, 1)$ and the tangent at C as $y = 0$. Then by Newton's diagram, or otherwise, it is readily seen that the coefficients of

$$z^4,\ z^3x,\ z^3y,\ z^2x^2,\ z^2xy,\ zx^3$$

in the equation of the quartic are zero. Hence the equation takes the form

$$z^2y^2 + 2zyu_2 + u_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i}),$$

where u_2 and u_4 are homogeneous of degree 2 and 4 in x, y.

The polar cubic of C with respect to the quartic is

$$y(zy + u_2) = 0.$$

Hence, if u_2 does not contain y as a factor,

The points of contact of the four tangents to a quartic from a tacnode C, other than the tangents at C, lie on a conic touching the quartic at C.

The conics of closest contact to the two branches of the quartic at C meet in two points at C and in two other points joined by a real line. Let this line meet $y = 0$ at $A\,(1, 0, 0)$, and let the point of contact of the second tangent from A to the conic $zy + u_2 = 0$ be $B\,(0, 1, 0)$. Then with a proper choice of homogeneous coordinates, we have $u_2 \equiv x^2$ and equation (i) becomes

$$(yz+x^2)^2 = (1-m)\,(x^4+p_1x^3y+p_2x^2y^2+p_3xy^3+p_4y^4) \quad . \text{ (ii)},$$

or, factorizing the right-hand side of (ii),

$$(yz+x^2)^2 = (1-m)\,(x+ay)\,(x+by)\,(x+cy)\,(x+dy) \quad . \text{ (iii)},$$

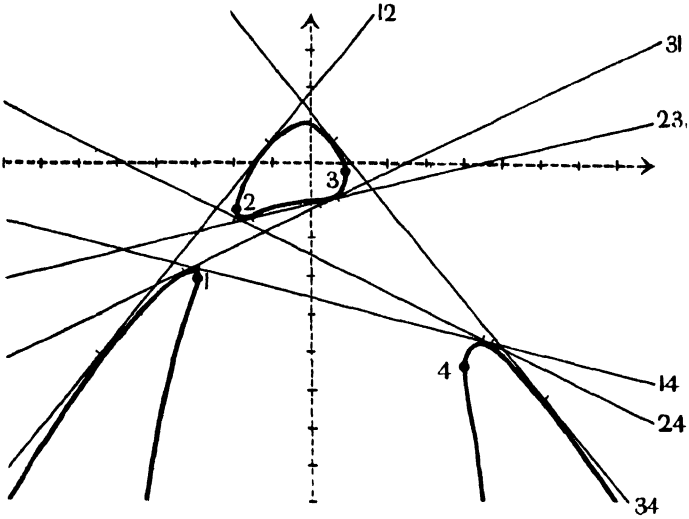

Fig. 9.
$16\,(3y+x^2)^2 = 7\,(x+3)\,(x+2)\,(x-1)\,(x-4)$.

the lines

$$x+ay = 0, \quad x+by = 0, \quad x+cy = 0, \quad x+dy = 0$$

being the tangents from C to the curve which touch at

$$C_1\,(a, -1, a^2), \quad C_2\,(b, -1, b^2), \quad C_3\,(c, -1, c^2), \quad C_4\,(d, -1, d^2).$$

The conics

$$yz+x^2 = \pm\sqrt{(1-m)}\,\{x^2+\tfrac{1}{2}p_1xy+(\tfrac{1}{2}p_2-\tfrac{1}{8}p_1^2)\,y^2\}$$

are the conics of closest contact with the quartic at C; for, squaring this equation and subtracting from (ii), we see that either conic meets the quartic in seven points at C. Since these conics have $y = 0$ and a line through A as a pair of common chords, we must have

$$p_1 \equiv a+b+c+d = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}.$$

Writing

$$\alpha = \tfrac{1}{2}(1+\sqrt{m}), \quad \beta = \tfrac{1}{2}(1-\sqrt{m})$$

we find that the line b_{12} (or b_{21}) .

$$\{\alpha(a+b)+\beta(c+d)\}\,x+(\alpha ab+\beta cd)\,y-z=0 \quad . \quad . \quad \text{(v)}$$

is a bitangent, touching where

$$\alpha(x+ay)(x+by)=\beta(x+cy)(x+dy).$$

This may be seen by writing (v) in the form

$$yz+x^2=\alpha(x+ay)(x+by)+\beta(x+cy)(x+dy)$$

and then squaring this last equation and subtracting from (iii).

Interchanging α and β we have another bitangent b_{34}.

We have thus three pairs of bitangents, excluding the tangent at C, namely b_{14} and b_{23}, b_{24} and b_{31}, b_{34} and b_{12}.

The bitangents are shown in Fig. 9 for the quartic *

$$16(3y+x^2)^2=7(x+3)(x+2)(x-1)(x-4).$$

Their equations are

$$\left.\begin{matrix}3x+12y+43=0\\3x-12y-13=0\end{matrix}\right\},\quad \left.\begin{matrix}12x+24y+59=0\\12x-24y-29=0\end{matrix}\right\},\quad \left.\begin{matrix}15x+12y-17=0\\15x-12y+23=0\end{matrix}\right\}$$

The points C_1, C_2, C_3, C_4 are

$$(-3,\,-3),\quad (-2,\,-\tfrac{4}{3}),\quad (1,\,-\tfrac{1}{3}),\quad (4,\,-\tfrac{16}{3}).$$

Ex. 1. C_1, C_2 and the points of contact of b_{12} lie on a conic touching the 4-ic at C.

[On $yz+x^2=2\alpha(x+ay)(x+by)$.]

Ex. 2. C_3, C_4 and the points of contact of b_{12} lie on a conic touching the 4-ic at C.

[On $yz+x^2=2\beta(x+cy)(x+dy)$.]

Ex. 3. Any pair of bitangents divides CA harmonically.

Ex. 4. Of the vertices of the quadrilateral formed by b_{12}, b_{34} and b_{14}, b_{23} two are the intersections of C_1C_2 and C_3C_4, C_1C_4 and C_2C_3. The other four lie on a conic touching the 4-ic at C and dividing AB harmonically.

[On $4(yz+x^2)=(1-m)\{2x+(a+c)y\}\{2x+(b+d)y\}$.]

Ex. 5. The bitangent b_{13} meets the 4-ic, b_{12} and b_{34}, b_{14} and b_{23} in an involution with a double point on CA.

Ex. 6. The intersections of each pair of bitangents lie on the same conic through A touching BC at C.

[On $p_3y^2+2p_2xy=4zx \qquad (p_1=0)$.]

Ex. 7. The points of contact of b_{12}, b_{34} lie on a conic touching the 4-ic at C.

[On $yz+mx^2+\tfrac{1}{2}(m-1)(ab+cd)y^2=0$.]

---

* For a more detailed discussion of the tacnodal quartic see *Messenger Math.*, xlvii (1917), p. 88. In Fig. 9 the suffixes only of C_1, b_{12}, &c., are given.

Ex. 8. If in § 14 u_2 has y as a factor, the curvatures of the branches touching at C are equal and opposite. In this case the points of contact of the four tangents from C are collinear. The equation of the quartic may be put in the form

$$My^2z^2+(x+ay)(x+by)(x+cy)(x+dy) = 0,$$

and the equation of the bitangent b_{12} in the form

$$2\sqrt{M}z = (a+b-c-d)x+(ab-cd)y = 0.$$

[The quartic can be projected into a symmetrical curve, and its properties deduced from this symmetry or by modification of Ex. 1 to 7.]

Ex. 9. Obtain properties of the tacnodal quartic by quadratic transformation.

[We can transform § 14 (iii) into the quartic of Ex. 8 by replacing z by $z-x^2/y$. Or we can transform it into the cubic of Ch. XVI, § 8, by replacing x by $x(y-ax)/y$, y by x^2/y, z by $z-(y-ax)^2/y$.]

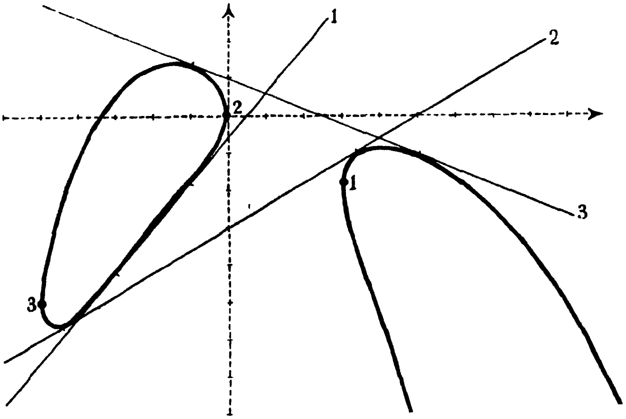

Fig. 10.
$(5y+x^2)^2 = 4x(x-3)(x+5)$.

§ 15. Quartics with a Rhamphoid Cusp.*

Ex. 1. If a quartic has a rhamphoid cusp C, the points of contact C_1, C_2, C_3 of the tangents from the cusp lie on a conic osculating the curve at C.

[Write down the polar cubic of C. See Ch. III, § 8, Ex. 2.]

Ex. 2. The equation of a quartic with a rhamphoid cusp can be put in the form $(yz+x^2)^2 = 4y(x+ay)(x+by)(x+cy)$.

The bitangents b_1, b_2, b_3 are $(b+c+1)x+(a+bc)y-z = 0$, &c.; C_1 is $(a, -1, a^2)$, &c.

[CA is the tangent at C, and AB touches the conic of Ex. 1 at B.]

---

* In Fig. 10 C is $(0, \infty)$ and CA the line at infinity. C_1, C_2, C_3 are $(3, -\frac{9}{5})$, $(0, 0)$, $(5, -5)$; b_1, b_2, b_3 are $6x = 5y+3$, $3x = 5y+15$, $2x+5y = 5$. The equation of the curve is $(5y+x^2)^2 = 4x(x-3)(x+5)$. In the figure C_1, b_1, &c., are represented by their suffixes only.

Ex. 3. The intersections of AC with b_1, b_2, b_3 form together with C a range of the same cross-ratio as the pencil $C(C_1 C_2 C_3 A)$.

Ex. 4. The points of contact of b_1 and C_1 lie on a conic osculating the quartic at C.

$[yz + x^2 = 2y(x + ay).]$

Ex. 5. The line joining the intersection of b_2 and b_3 to the intersection of b_1 and AC passes through C_1.

Ex. 6. The points of contact of b_2 and b_3 lie on a conic through C, C_2, C_3.

Ex. 7. There is no loss of generality in taking $a = 0$ in Ex. 2.

Obtain properties of the curve by transforming it quadratically into a cubic in this case.

[(i) Take B as C_1 on the conic of Ex. 1. (ii) Replace y by x^2/y and z by $z - y$.]

CHAPTER XIX

NON-SINGULAR QUARTICS

§ 1. Non-singular Quartics.

IN this chapter we consider the non-singular quartic without double or triple point. It is of class 12, has twenty-eight bitangents, and twenty-four inflexions.

Of the geometry of the inflexions practically nothing is known, except such properties as are special cases of theorems relating to the inflexions of a curve of any degree.

The properties of the twenty-eight bitangents have been worked out in very great detail. We cannot spare space for more than the most interesting of these results. The reader who wishes to pursue the matter further may consult Weber's *Algebra*, II, §§ 112–122; Miller, Blickfeldt, and Dickson's *Finite Groups*, ch. xix; and the references in *Crelle*, xcix, p. 275; cvii, p. 1; cxxii, p. 209; *Proc. London Math. Soc.*, II. ix (1910), pp. 145, 205; &c., &c.

We shall return to the subject in Ch. XX, § 8.

Ex. 1. Find the inflexional tangents and bitangents of

$$x^4 + y^4 + z^4 = 0.$$

[$y = (-1)^{\frac{1}{4}} z$, &c., are twelve undulatory tangents, counting as twenty-four inflexional and twelve bitangents.

The remaining bitangents are $x + 1^{\frac{1}{4}} y + 1^{\frac{1}{4}} z = 0$, &c.]

Ex. 2. The bitangents of

$$ax^4 + by^4 + cz^4 + 2fy^2z^2 + 2gz^2x^2 + 2hx^2y^2 = 0$$

are $Bz^4 - 2Fy^2z^2 + Cy^4 = 0$ and eight similar bitangents,

$$(F - \sqrt{BC})^{\frac{1}{2}} x \pm (G - \sqrt{CA})^{\frac{1}{2}} y \pm (H - \sqrt{AB})^{\frac{1}{2}} z = 0,$$

and the same with two of the $-\sqrt{}$ changed into $+\sqrt{}$.

[The notation is that of Ch. XVII, § 3. See Mathews, *Proc. London Math. Soc.*, xxii (1891), p. 173.]

Ex. 3. The equation of any non-singular quartic can be put in the form

$$ax^4 + by^4 + cz^4 + fyz(2y^2 - 3yz + 2z^2) + gzx(2z^2 - 3zx + 2x^2)$$
$$+ hxy(2x^2 - 3xy + 2y^2) + px^2(3yz - xy - xz) + qy^2(3zx - yz - yx)$$
$$+ rz^2(3xy - zx - zy) = 0.$$

[The polar cubic of (1, 1, 1) is canonical.]

§ 2. Bitangents of Non-singular Quartics.

Suppose that any two of the twenty-eight bitangents of a non-singular quartic are taken as $x = 0$, $y = 0$. Let its equation arranged in descending powers of z be

$$z^4 + 2z^3(ax+by) + \ldots = 0.$$

When we put $y = 0$, the left-hand side reduces to a perfect square, say $(z^2+azx+cx^2)^2$; and when we put $x = 0$, it reduces to a perfect square, say $(z^2+bzy+dy^2)^2$.

The equation must therefore evidently be of the form

$$z^4 + xyu + 2az^3x + (a^2+2c)z^2x^2 + 2aczx^3 + c^2x^4$$
$$+ 2bz^3y + (b^2+2d)z^2y^2 + 2bdzy^3 + d^2y^4 = 0,$$

where $u = 0$ is a conic.

This may be written

$$xyU = V^2 \quad \ldots\ldots\ldots \quad \text{(i)},$$

where

$$U \equiv 2abz^2 + 2(bcx+ady)z + 2cdxy - u,$$
$$V \equiv z^2 + (ax+by)z + (cx^2+dy^2);$$

so that $U = 0$ and $V = 0$ are conics.

Now (i) may be written

$$xy(U+2kV+k^2xy) = (V+kxy)^2.$$

Choose k so that $U+2kV+k^2xy$ has a pair of factors p, q linear in x, y, z. The condition that $U+2kV+k^2xy$ should factorize is readily seen to be of degree 5 in k, so that k may be chosen in general in five ways. The equation of the quartic then takes the form

$$xypq = W^2 \quad \ldots\ldots\ldots \quad \text{(ii)},$$

where $W \equiv V+kxy$.

It is evident that $p = 0$, $q = 0$ are also bitangents, and that the conic $W = 0$ passes through the eight points of contact of the four bitangents $x = 0$, $y = 0$, $p = 0$, $q = 0$.

Hence:

Through the four points of contact of two bitangents of a non-singular quartic pass five conics each of which passes through the points of contact of two more bitangents.

Since k can be chosen in five ways, we obtain thus six pairs of bitangents, namely $x = 0$, $y = 0$ and five pairs such as $p = 0$, $q = 0$. Now these six pairs have the property that the eight points of contact of *any* two of these pairs lie on a conic.

For suppose that any two such pairs are $p=0$, $q=0$ and $r=0$, $s=0$. Then the quartic is $f=0$, where

$$f \equiv xypq - W^2 \equiv xyrs - Z^2;$$

$W=0$ and $Z=0$ being conics.

We deduce

$$xy(pq-rs) \equiv (W-Z)(W+Z).$$

Now we cannot have x a factor of $W-Z$ and y of $W+Z$, or vice versa. For otherwise

$$W \equiv \tfrac{1}{2}(W+Z) + \tfrac{1}{2}(W-Z)$$

will vanish when $x=y=0$, and the quartic will go through the point $(0, 0, 1)$; which is impossible, since $x=0$ and $y=0$ only meet the curve in their points of contact.

Hence we must have

$$xy = \mu(W \mp Z) \text{ and } pq-rs = \mu^{-1}(W \pm Z),$$

where μ is a constant.

Now a bitangent is not altered by multiplying its equation through by a constant. Hence there is no loss of generality in supposing $\mu=1$ and the upper signs taken in the ambiguity.

Then we have

$$\begin{aligned} -4f \equiv -4xy\,pq + 4W^2 &\equiv -4xy\,pq + (xy+pq-rs)^2 \\ &\equiv x^2y^2 + p^2q^2 + r^2s^2 - 2xy\,pq - 2xy\,rs - 2pq\,rs. \end{aligned}$$

The equation of the quartic is therefore

$$(xy)^{\frac{1}{2}} + (pq)^{\frac{1}{2}} + (rs)^{\frac{1}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iii)}.$$

The symmetry of this result shows that the eight points of contact of $p=0$, $q=0$ and $r=0$, $s=0$ lie on a conic, which is the required result.

Three bitangents of a quartic are called *syzygetic* or *asyzygetic*, according as their six points of contact do or do not lie on a conic. Similarly *four* bitangents will be called *syzygetic* or *asyzygetic* according as their eight points of contact do or do not lie on a conic.

Consider now the pair of bitangents $x=0$, $p=0$. There are five pairs of bitangents such that the points of contact of any pair lie on a conic through the points of contact of $x=0$ and $p=0$. One such pair is $y=0$ and $q=0$. Let another pair be $t=0$ and $u=0$.

We have shown that, if $W=0$ is any conic through the points of contact of $x=0$, $y=0$, $p=0$, $q=0$, $2W$ is of the form $xy/\mu + \mu(pq-rs)$; similarly it is of the form

$$xp/\nu + \nu(yq-tu).$$

Comparing these expressions for $2W$ we get

$$\left(\frac{x}{\mu\nu}-q\right)(\nu y-\mu p)\equiv\mu rs-\nu tu.$$

Therefore the line through the intersections of $r=0$, $t=0$ and $s=0$, $u=0$ passes through the intersection of either $x=0$, $q=0$ or $y=0$, $p=0$.

Hence:

The 378 intersections of the bitangents lie by threes on straight lines.

Ex. 1. A conic through the points of contact of two bitangents meets the curve again in A, B, C, D. Show that a conic can be drawn touching the quartic at A, B, C, D.

[If the quartic is $xyU=V^2$ and the conic is $V+kxy=0$, the required conic is $U+2kV+k^2xy=0$. For the quartic may be written

$$xy(U+2kV+k^2xy)=(V+kxy)^2.]$$

Ex. 2. Two conics are drawn through the points of contact of two bitangents. Show that their eight other intersections with the quartic lie on a conic.

[Taking the conics as $V+k_1xy=0$ and $V+k_2xy=0$, the required conic is $U+(k_1+k_2)V+k_1k_2xy=0$. The preceding example is a limiting case of this. See also Ch. XII, § 6, Ex. 5.]

Ex. 3. If ABC is the triangle formed by three real bitangents whose points of contact are P_1 and P_2, Q_1 and Q_2, R_1 and R_2, then

$$BP_1 \cdot BP_2 \cdot CQ_1 \cdot CQ_2 \cdot AR_1 \cdot AR_2=\pm BR_1 \cdot BR_2 \cdot AQ_1 \cdot AQ_2 \cdot CP_1 \cdot CP_2;$$

the upper or lower sign being taken according as the bitangents are syzygetic or asyzygetic. An even or odd number of the involutions (BC, P_1P_2), (CA, Q_1Q_2), (AB, R_1R_2) are overlapping in the two cases respectively.

[Use Ch. I, § 6, Ex. 2. Compare Ch. II, § 3, Ex. 19.]

Ex. 4. If four bitangents, not all concurrent, are such that any three are syzygetic, then all four bitangents are syzygetic.

[Project the points of contact of one of the bitangents into the circular points, and remember that a pair of common chords of a circle and conic are equally inclined to the axes of the conic.]

Ex. 5. Three concurrent bitangents are syzygetic.

[By a proper choice of coordinates the bitangents become

$$x^2+2hxy+y^2=0 \quad\text{and}\quad y=0.$$

The equation of the quartic is $(x^2+2hxy+y^2)U=V^2$, and, if $y=0$ is a bitangent, we find that

$$U+2kV+k^2(x^2+2hxy+y^2)$$

will have y as a factor for a suitable value of k.]

Ex. 6. Four concurrent bitangents are syzygetic.

[As in Ex. 5, taking $x=0$ as the fourth bitangent.]

Ex. 7. The four common tangents to two ovals of a quartic are syzygetic.

[Use Ex. 3.]

Ex. 8. If the sides of the triangle of reference are three bitangents of a quartic, the equation of the quartic can be put in one of the two forms

$$p^2x^2+q^2y^2+r^2z^2 \pm 2qryz \pm 2rpzx \pm 2pqxy = xyzw,$$

where p, q, r, w are linear functions of x, y, z.

If the upper signs are taken, the bitangents are syzygetic and $p\,dx+q\,dy+r\,dz$ is an exact differential.

If the lower signs are taken, the bitangents are asyzygetic.

If $p\,dx+q\,dy+r\,dz$ is an exact differential in this latter case, the bitangents are the diagonals of a quadrilateral whose vertices are their points of contact.

[Suppose that when we put $x=0$, $y=0$, $z=0$, the equation reduces to $(by^2+2fyz+cz^2)^2$, $(\pm cz^2+2gzx+ax^2)^2$, $(\pm ax^2+2hxy+by^2)^2$, respectively. The signs of a, b, c may be changed without loss of generality; and we readily obtain in the two possible cases

$$p = ax+hy+gz, \qquad q = hx+by+fz, \qquad r = gx+fy+cz,$$

or

$$p = ax+hy-gz, \qquad q = -hx+by+fz, \qquad r = gx-fy+cz.$$

If in the second case $p\,dx+q\,dy+r\,dz$ is an exact differential,

$$f=g=h=0.$$

See Humbert, *Liouville's Journal*, IV. vi (1890), p. 423.]

§ 3. Steiner's Complex.

Six pairs of bitangents such that any two pairs are syzygetic are said to form a *Steiner's Complex*.*

We shall denote the pairs by a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6. The points of contact of a_1 will be denoted by A_1 and A_1', &c.

Either bitangent of any pair of a complex is met by the other five pairs in an involution. The points of contact of the given bitangent also belong to the involution.

Let a_6 be the bitangent. It meets all conics through A_1, A_1', B_1, B_1' in an involution. The lines a_1 and b_1, the conic $A_1A_1'B_1B_1'A_6A_6'$, and the conic $A_1A_1'B_1B_1'A_2A_2'B_2B_2'$ are three such conics. Suppose a_6 meets this last conic in P and Q, and meets a_1, b_1, a_2, b_2 in α_1, β_1, α_2, β_2. Then

$$(PQ,\ \alpha_1\beta_1,\ A_6A_6')$$

is an involution, and similarly $(PQ,\ \alpha_2\beta_2,\ A_6A_6')$ is an involution. Hence $(A_6A_6',\ \alpha_1\beta_1,\ \alpha_2\beta_2)$ is an involution, and similarly for the intersections of a_6 with a_3 and b_3, &c.†

The twelve bitangents of a complex touch a curve of the third class.

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* Or *Steiner's Group* or *Steiner's Set* or *Double-six*. We shall say 'a complex'.

† Another proof is the following: Take $x=0$, $y=0$ as a_6b_6. Then taking § 2 (i) as the equation of the quartic, the line-pairs a_1b_1, ..., a_5b_5 are included among the conics $U+2kV+k^2xy=0$. But it is at once proved that these conics cut $x=0$ in an involution.

For it is readily proved that the envelope of a line divided by three line-pairs in an involution is a curve of the third class * touching the lines in question.

Three bitangents of a complex, no two of which belong to the same pair, are asyzygetic.

Taking the bitangents as the sides of the triangle of reference, the equation of the quartic takes the form

$$(l_1x^2+m_1xy+n_1xz)^{\frac{1}{2}}+(l_2yx+m_2y^2+n_2yz)^{\frac{1}{2}}$$
$$+(l_3zx+m_3zy+n_3z^2)^{\frac{1}{2}}=0$$

by § 2 (iii).

The intersections of this with the sides of the triangle of reference are at once found, and can readily be shown not to lie on a conic.†

The six intersections of each pair of a complex lie on a conic.

This is equivalent to proving that the six centres of the six line-pairs included in the family of conics

$$S+2kS'+k^2S''=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)},$$

where

$$S\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy, \quad S'\equiv a'x^2+\ldots,$$
$$S''=a''x^2+\ldots,$$

lie on a conic.

For the sake of symmetry we have replaced the U, V, and xy of § 2 by S, S', S''.

If $S=0$ is a line-pair, its centre (x, y, z) is given by the equations

$$ax+hy+gz=0, \quad hx+by+fz=0, \quad gx+fy+cz=0,$$

which give

$$x^2/A=y^2/B=z^2/C=yz/F=zx/G=xy/H,$$

where $\qquad A=bc-f^2, \quad F=gh-af, \quad$ &c.

Suppose now (i) is a line-pair when $k=k_1, k_2, \ldots,$ or k_6; and that the quantities A, B, ... become A_i, B_i, ... when we replace S by $S+2k_iS'+k_i^2S''$ $(i=1, 2, \ldots, 6)$.

The conditions that the centres of the line-pairs lie on a conic

$$\mathbf{a}x^2+\mathbf{b}y^2+\mathbf{c}z^2+2\mathbf{f}yz+2\mathbf{g}zx+2\mathbf{h}xy=0$$

are

$$\mathbf{a}A_i+\mathbf{b}B_i+\mathbf{c}C_i+2\mathbf{f}F_i+2\mathbf{g}G_i+2\mathbf{h}H_i=0, \quad (i=1, \ldots, 6).$$

---

* See Ch. XV, § 5, Ex. 2.

† By Carnot's theorem, or otherwise, the points $x=b_1y^2+c_1z^2+2\alpha yz=0$, $y=c_2z^2+a_2x^2+2\beta zx=0$, $z=a_3x^2+b_3y^2+2\gamma xy=0$ lie on a conic, if $a_3b_1c_2=a_2b_3c_1$. In this case we get $l_1m_2n_3=0$, as the condition that the intersections should lie on a conic, which is excluded.

Therefore the six centres lie on a conic, if the determinant Δ of the sixth order whose rows are

$$A_i,\ B_i,\ C_i,\ F_i,\ G_i,\ H_i\ \ (i = 1, 2, \ldots, 6)$$

vanishes.

But this is the case, for $\Delta = 0$ considered as an equation in k_1 is of the fourth degree, and has five roots

$$k_1 = k_2,\ k_3,\ k_4,\ k_5, \text{ and } k_6.$$

Hence Δ vanishes identically.*

There are sixty-three complexes of bitangents.

For any two bitangents determine a complex. There are ${}^{28}C_2 = 378$ such pairs. Any one of the six pairs of a complex determines the same complex. Hence the number of complexes is $378 \div 6 = 63$.

Ex. 1. Any non-singular quartic is the envelope of the polar conic of any point P on a certain fixed conic with respect to a fixed cubic.

[The quartic is the envelope of the conic $U + 2kV + k^2xy = 0$ for varying k. The cubic is a curve of which the Jacobian of $U = 0$, $V = 0$, $xy = 0$ is the Hessian.]

Ex. 2. The pairs of bitangents of a complex are the polar conics of P in Ex. 1, if P is at an intersection of the fixed conic with the Hessian. The curve of the third class touched by the bitangents of the complex is the corresponding Cayleyan.

[See Ch. XV, § 4.]

Ex. 3. Use Ch. XVI, § 6, Ex. 25, to deduce from Ex. 2 that the intersections of pairs of bitangents of a complex lie on a conic.

§ 4. Relations between Complexes.

We have seen in § 2 that from any two bitangents a single complex may be obtained. In the complex whose pairs are a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6 take any two bitangents not forming a pair, say a_1 and a_2. These determine a complex with the pairs a_1a_2, b_1b_2. The remaining bitangents of this new complex are not in the original complex. For by the result that three bitangents of a complex are asyzygetic, if no two belong to the same pair, no one of $a_3, b_3, \ldots, a_6, b_6$ can be syzygetic to both a_1a_2 and b_1b_2. Similarly a_1 and b_2 determine a complex containing the pair a_2b_1; and no two of the three complexes thus determined by a_1b_1, a_1a_2, a_1b_2 respectively have a bitangent in common other than a_1, b_1, a_2, b_2 which are common to all three. We call three such complexes a *complex-triple*. Between them they contain all the twenty-eight bitangents; four of the bitangents lying in all three complexes, and the other twenty-four lying in one of the complexes only.

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* This proof is due to Baker, *Proc. London Math. Soc.*, II. ix (1910), p. 148.

Two pairs of the given complex such as a_1a_2, b_1b_2 can be selected in ${}^6C_2 = 15$ ways. Hence each complex is a member of fifteen complex-triples including in all thirty-one complexes.

Returning now to the given complex with pairs

$$a_1b_1,\ a_2b_2,\ a_3b_3,\ a_4b_4,\ a_5b_5,\ a_6b_6;$$

take any bitangent c_1 not in the complex. We get two new complexes determined by a_1c_1 and b_1c_1. Since c_1 can be chosen in sixteen ways, we get thus thirty-two more complexes, which with the thirty-one complexes just mentioned make up the total of sixty-three.

We shall show that the complexes determined by a_1c_1, b_1c_1 have six bitangents in common, i.e. their pairs are of the form

$$a_1c_1,\ a_2c_2,\ a_3c_3,\ a_4c_4,\ a_5c_5,\ a_6c_6;$$
$$b_1c_1,\ b_2c_2,\ b_3c_3,\ b_4c_4,\ b_5c_5,\ b_6c_6.$$

For the bitangent c_1 must lie in the complex determined by a_1a_2 or in that determined by a_1b_2. There is no loss of generality in supposing it to be in the former; for the given complex determined by a_1b_1 is not altered by interchange of a_2 and b_2. Suppose then that a_1a_2, c_1c_2 are pairs of a complex, and consider the three complexes with pairs

$$a_1a_2,\ b_1b_2,\ c_1c_2,\ \ldots;$$
$$a_1c_1,\ a_2c_2,\ \ldots,\ \ldots;$$
$$a_1c_2,\ a_2c_1,\ \ldots,\ \ldots.$$

These form a complex-triple and therefore contain all the bitangents, and in particular a_3 and b_3. Neither of them is in the first complex of the triple. They cannot both be in the second (or both in the third) complex. For they are not a pair of this complex, and three bitangents of the complex no two of which are a pair are asyzygetic, whereas a_1, a_3, b_3 are syzygetic. Since a_3 and b_3 are interchangeable in the given complex, we may without loss of generality suppose a_3 to be in the second (and b_3 in the third) complex of the triple. Let then a_1c_1. a_2c_2, a_3c_3 be three pairs of the second complex. Now the complex with the pairs a_2a_3, c_2c_3 contains the pair b_2b_3, since a_2, a_3, b_2, b_3 are syzygetic. Hence the complex with the pairs b_1c_1, b_2c_2 contains the pair b_3c_3. This argument is at once extended, and the result proved.

Summing up we have

Two complexes have either four syzygetic bitangents in common and form with another complex containing the same four bitangents a complex-triple including all the twenty-eight *bitangents; or else they have six bitangents in common no three of which are asyzygetic, and form a complex-pair of eighteen bitangents.*

The complexes formed by

$$a_1b_1,\ a_2b_2,\ a_3b_3,\ a_4b_4,\ a_5b_5,\ a_6b_6;$$
$$a_1c_1,\ a_2c_2,\ a_3c_3,\ a_4c_4,\ a_5c_5,\ a_6c_6;$$
$$b_1c_1,\ b_2c_2,\ b_3c_3,\ b_4c_4,\ b_5c_5,\ b_6c_6$$

are such that any two have six asyzygetic bitangents in common. The seven bitangents $a_1, a_2, a_3, a_4, a_5, c_6, b_6$ are such that no three of them are syzygetic Such a set of bitangents is called an 'Aronhold Seven'. Another Aronhold Seven is $b_1, b_2, b_3, b_4, b_5, a_6, c_6$.

§ 5. The Hessian Notation.

In the three complexes at the end of § 4 alter the notation by interchanging a_6 and b_6, and by writing c for c_6. They become

$$a_1b_1,\ a_2b_2,\ a_3b_3,\ a_4b_4,\ a_5b_5,\ a_6b_6;$$
$$a_1c_1,\ a_2c_2,\ a_3c_3,\ a_4c_4,\ a_5c_5,\ b_6c;$$
$$b_1c_1,\ b_2c_2,\ b_3c_3,\ b_4c_4,\ b_5c_5,\ a_6c.$$

Then $a_1, a_2, a_3, a_4, a_5, a_6, c$ is an Aronhold Seven; and so is $b_1, b_2, b_3, b_4, b_5, b_6, c$.

Denote the bitangents $a_1, a_2, a_3, a_4, a_5, a_6$ by the symbols 18, 28, 38, 48, 58, 68 or 81, 82, 83, 84, 85, 86, and the bitangents $b_1, b_2, b_3, b_4, b_5, b_6$ by 17, 27, 37, 47, 57, 67 or 71, 72, 73, 74, 75, 76. Denote c by 78 or 87.

Now c must occur in one complex of the triple

$$a_ib_i,\ a_jb_j,\ \ldots;\quad a_ia_j,\ b_ib_j,\ \ldots;\quad a_ib_j,\ a_jb_i,\ \ldots;$$

i and j being any two of the digits 1, 2, 3, 4, 5, 6.

It does not occur in the first by definition, and cannot occur in the second as it is asyzygetic to a_ia_j. Hence it must occur in the third. The bitangent paired to it in this complex is denoted by ij or ji.

There are ${}^8C_2 = 28$ symbols such as $ij \equiv ji$. Hence we have a notation which will include all the twenty-eight bitangents. We note that the bitangents ij and rs cannot be the same. For, if they were,

$$c\,ij,\ a_ib_j,\ a_jb_i,\ a_rb_s,\ a_sb_r,\ \ldots$$

is a complex; therefore

$$a_ia_r,\ b_jb_s,\ a_ja_s,\ \ldots$$

is a complex, contrary to the supposition that a_i, a_j, a_r are asyzygetic.

The notation is not symmetrical, but it is perhaps as convenient as any we can find.

We may visualize* the notation by considering the quartic $fg+\epsilon=0$, where ϵ is a small constant, and $f=0$, $g=0$ are equal ellipses of small eccentricity and nearly concentric, the major axis of one being parallel to the minor axis of the other. The quartic consists of four portions each lying inside one ellipse and outside the other; see Fig. 1.

These portions are narrow ovals each touched by one bitangent, while any two ovals have four tangents in common.

Hence the quartic has twenty-eight real bitangents each touching the curve in points which lie very close to two of the points labelled 1, 2, 3, 4, 5, 6, 7, 8 in the figure.

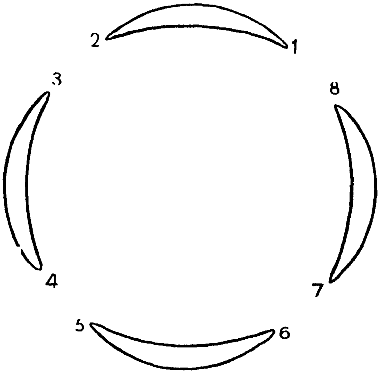

Fig. 1.

If we proceed to the limit in which the ovals are indefinitely narrow, we may consider the bitangents as the twenty-eight lines joining in pairs the eight points 1, 2, ..., 8 (Fig. 2).

The bitangents are now denoted by the symbols 12 or 21, 13 or 31, &c. We shall show that this is the same notation for the bitangents as that before derived at the beginning of this section.

The criterion of § 2, Ex. 3, applied to Fig. 1 will readily show that three bitangents are syzygetic, if (i) their symbols in Fig. 2 involve six distinct digits, e.g. 12, 34, 56; (ii) their symbols involve two digits twice and two digits once, e.g. 12, 23, 34.

It follows from § 2, Ex. 4, that four bitangents are syzygetic, if their symbols are of one of the two types 12, 34, 56, 78 or 12, 23, 34, 41.

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* Fontené, *Bull. de la Soc. Math. de France*, xxvii, p. 229.

In Fig. 2 the four bitangents either join four distinct pairs of points or else are the sides of a quadrilateral.

It follows at once that we have complexes of the two types

$$18\ 17,\ 28\ 27,\ 38\ 37,\ 48\ 47,\ 58\ 57,\ 68\ 67;$$
$$\text{or}\ \ 14\ 23,\ 24\ 31,\ 34\ 12,\ 58\ 67,\ 68\ 75,\ 78\ 56.$$

The first type contains ${}^8C_2 \equiv 28$ complexes and the second $\frac{1}{2}\,.\,{}^8C_4 \equiv 35$ complexes. Hence we have thus all possible sixty-three complexes; and then, retracing our steps, we see that we have given above every possible set of three or four syzygetic bitangents. Moreover, since we have just shown that

$$i8\,j7,\ i7\,j8,\ ij\ 78,\ \ldots$$

is a complex in Fig. 1, we see that the notation for the bitangents obtained at the beginning of the section agrees with that obtained from Fig. 2.

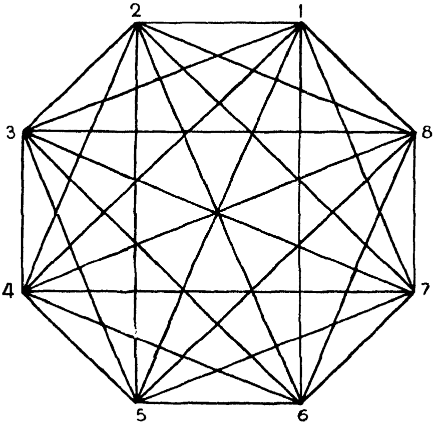


Fig. 2.

We can deduce the fact that three bitangents of the type pq, qr, rs or three of the type pq, rs, tu are syzygetic, from the original definition without the use of Fig. 1.*

We begin with the type pq, qr, rs.

First take the case in which two of p, q, r, s are 7 and 8. We have the possible types †

$$j7,\ 78,\ 8i\ \text{and}\ 87,\ 7j,\ ji\ \text{and}\ 7j,\ j8,\ 8i\ \text{and}\ 7j,\ ji,\ i8.$$

---

* The figure only gives a special quartic, and therefore must not be relied upon for a proof, though it assists us to visualize the results obtained in this chapter.

† i, j, k, l, ... denote any of the integers 1, 2, 3, 4, 5, 6.

These are all syzygetic, for

$$i8\,j7,\ i7\,j8,\ ij\,78,\ \ldots$$

is a complex by definition.

Now take the cases in which one of p, q, r, s is 7 or 8. We have the two possible types

$$i8,\ 8j,\ jk \text{ and } 8i,\ ik,\ kj.$$

For the first type we note that jk is in one of the triple of complexes

$$\begin{array}{llllll} 18\ 17, & 28\ 27, & 38\ 37, & 48\ 47, & 58\ 57, & 68\ 67; \\ i8\ j7, & i7\ j8, & ij\ 78, & \ldots & & ; \\ i8\ j8, & i7\ j7, & \ldots & & & . \end{array}$$

It is not in the first. It is not in the second; for $j7$, 78, jk are syzygetic. Therefore it is in the third; which makes $i8$, $8j$, jk syzygetic.

It follows that $i8$ is in the complex $j8\,jk, \ldots$. But, changing the notation, $j8$, $8i$, ik are syzygetic. Hence

$$j8\,jk,\ i8\,ik,\ \ldots$$

is a complex, for otherwise we should have three bitangents in different pairs of a complex syzygetic. Therefore for any value of k

$$18\ 1k,\ 28\ 2k,\ \ldots,\ 78\ 7k$$

is a complex.

It follows that the other type $8i$, ik, kj is also syzygetic; since this complex is

$$i8\,ik,\ j8\,jk,\ \ldots.$$

A similar process holds if we interchange 7 and 8.

Now take the type ij, jk, kl.

We have shown that

$$l8\,lk,\ 78\,7k,\ j8\,jk,\ \ldots \text{ and } 17\ 18,\ 27\ 28,\ 37\ 38,\ \ldots$$

are complexes, whence

$$kl\,kj,\ 8l\,8j,\ 7l\,7j,\ \ldots$$

is a complex.

Repetition of this argument shows that

$$1l\ 1j,\ 2l\ 2j,\ \ldots,\ 8l\ 8j$$

is a complex; and ij, jk, kl are syzygetic.

This exhausts all trios of the type pq, qr, rs.

We have now shown that there are complexes of the type

$$1p\ 1q,\ 2p\ 2q,\ \ldots,\ 8p\ 8q,$$

where p and q are any of the integers 1, 2, 3, 4, 5, 6, 7, 8.

Now consider three bitangents of the type *pq*, *rs*, *tu*; for instance, 12, 34, 56. Since 56 is in one complex of the triple

12 14, 32 34, 52 54, 62 64, 72 74, 82 84;
21 23, 41 43, 51 53, 61 63, 71 73, 81 83;
12 34, 14 32, ...,

and is not in the first two, it is in the third. Hence 12, 34, 56 are syzygetic.

This completes the proof that three bitangents of the type *pq*, *qr*, *rs* or *pq*, *rs*, *tu* are syzygetic. Then the proof that complexes are of two types; and that there are only two types of three or four syzygetic bitangents follows as before.

Ex. 1. Obtain all the 288 Aronhold Sevens of bitangents.

[Lines in Fig. 2 joining one point to the other seven; or lines forming a triangle together with the lines joining one of the remaining points to the other four.]

Ex. 2. How many sets of four syzygetic bitangents exist?

[105 + 210.]

Ex. 3. The bitangents c_1, c_2, c_3, c_4, c_5 of § 5 are 16, 26, 36, 46, 56.

§ 6. Lines on a Cubic Surface.

If we take any point O on a cubic surface as the point (0, 0, 0, 1), the equation of the surface in homogeneous coordinates is

$$f \equiv u_1 w^2 + 2u_2 w + u_3 = 0,$$

where u_r is homogeneous of degree r in x, y, z.

The tangent-lines from O to the surface have their points of contact on

$$\frac{\partial f}{\partial w} \equiv 2(wu_1 + u_2) = 0.*$$

Eliminating w between $f = 0$ and $\dfrac{\partial f}{\partial w} = 0$, we have the equation $u_1 u_3 = u_2^2$ for the intersection of the tangent-lines with the plane $w = 0$. This is a plane quartic with the intersection $u_1 = 0$ of $w = 0$ and the tangent-plane at O to the surface as a bitangent. The points of contact of this bitangent lie on $u_1 = 0$, $u_2 = 0$, i.e. on the inflexional tangents of the cubic at O.

Conversely, *any* non-singular quartic can be derived from a cubic surface in this manner; for the equation of any quartic can be put in the form $u_1 u_3 = u_2^2$, where $u_1 = 0$ is any bitangent.

---

* The proof is similar to that which shows that the points of contact of the tangents from O to a plane curve lie on the first polar curve of O.

This result enables us to deduce properties of the plane quartic from those of the cubic surface and vice versa.*

For the present let us denote the bitangent $w = 0$, $u_1 = 0$ by l. The plane joining O to any bitangent of the quartic, other than l, touches the surface in two points P, Q one on each of the lines joining O to the points of contact of the bitangent. Hence the curve of intersection of the plane with the surface has P and Q for nodes. But the curve is of degree 3, and must therefore degenerate into the line PQ and a conic.

Hence the projections from O on to $w = 0$ of the lines on the surface are the bitangents of the quartic other than l.

It follows that there are twenty-seven lines on a cubic surface.

In Ch. XX we shall show that either 4, 8, 16, or 28 of the bitangents of a non-singular quartic are real. Hence either 3, 7, 15, or 27 of the lines on a non-singular cubic surface are real.

There is evidently no loss of generality in supposing that the plane $w = 0$ passes through a line of the surface. Take the line as $w = z = 0$. The equation of the surface is

$$u_1 w^2 + 2Vw + Uz = 0,$$

where U and V are homogeneous of degree 2 in x, y, z.

The surface meets the plane $w = kz$ where

$$z\,(U + 2kV + k^2 u_1 z) = 0.$$

Hence the conics in which planes through the line $w = z = 0$ meet the surface again project from O into the family of conics

$$U + 2kV + k^2 u_1 z = 0.$$

By choosing k to make such a conic a line-pair, we obtain five pairs of bitangents of the quartic which together with the pair $w = z = 0$, $w = u_1 = 0$ make up a Steiner complex (§ 3). The plane $w = kz$ for such a value of k meets the surface in three lines which form a degenerate cubic with three nodes, at each of which the plane touches the surface. Hence:

Through any line m of a cubic surface five planes can be drawn each touching the surface in three points and meeting it again in a line-pair. The projections of these five line-pairs form with l and the projection of m a complex of bitangents of the quartic curve.

---

* The method is due to Geiser, *Math. Annalen*, I, p. 129.

The number of such triple tangent-planes is evidently

$$27 \times 5 \div 3 = 45.$$

The above result gives us all the twenty-seven complexes which contain l. We now proceed to find the other thirty-six complexes.

§ 7. Schläfli's Double-six.

Suppose that the line a_6 of a cubic surface is met by the five line-pairs (§ 6)

$$b_1c_1,\ b_2c_2,\ b_3c_3,\ b_4c_4,\ b_5c_5;$$

the planes $a_6b_1c_1$, &c., being triple tangent-planes of the surface.

No two of c_1, c_2, c_3 can meet. Suppose that c_4 meets the hyperboloid with c_1, c_2, c_3 as generators in two points. Through each of them passes a generator of the opposite family to c_1, c_2, c_3. One of these is a_6; let the other be b_6. Since b_6 meets c_1, c_2, c_3, c_4 in four points lying on the surface, b_6 is a line of the surface.

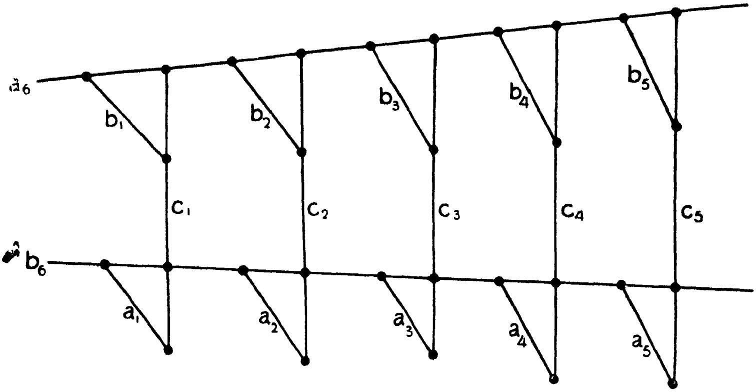


Fig. 3 (diagrammatic only).

The plane of $a_6c_5b_5$ meets the surface in the three lines a_6, c_5, b_5. Hence b_6 meets one of these lines. It cannot meet a_6; there is no loss of generality in supposing it to meet c_5.

Let the planes $b_6c_1, b_6c_2, b_6c_3, b_6c_4, b_6c_5$ meet the surface again in the lines a_1, a_2, a_3, a_4, a_5. Then it is at once seen that the configuration of twelve lines on the surface

$$\begin{Bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{Bmatrix}$$

is such that any line does *not* meet each of the lines in the same row or column in the array { }, but does meet the other five lines. For instance, a_1 meets b_2, b_3, b_4, b_5, b_6 only. Such a configuration is called a 'Schläfli's double-six'.

Since the triple tangent-planes through a_1 are

$$a_1 b_2,\ a_1 b_3,\ a_1 b_4,\ a_1 b_5,\ a_1 b_6,$$

it follows readily that we might have constructed the double-six starting from the pair $a_1 b_1$ instead of $a_6 b_6$. In other words, the double-six is symmetrical as regards its six pairs

$$a_1 b_1,\ a_2 b_2,\ a_3 b_3,\ a_4 b_4,\ a_5 b_5,\ a_6 b_6.$$

Denoting the bitangents into which the lines $a_1, b_1, c_1, \ldots$ project by the same symbols, we showed in § 6 that $a_1 b_6$, $a_6 b_1$ are pairs of the complex determined by l and c_1. Hence a_1, a_6, b_1, b_6 are syzygetic, or $a_1 b_1$ is a pair of the complex determined by $a_6 b_6$. Similarly for $a_2 b_2$, $a_3 b_3$, &c. Hence:

A double-six projects into a complex of bitangents.

We showed that each line on a cubic surface meets ten other lines, and therefore does not meet sixteen lines. Hence there are $27 \times 16 \div 2 = 216$ non-intersecting pairs of lines on the surface. Therefore, as each double-six may be determined by any one of its pairs, the number of double-sixes is $216 \div 6 = 36$. Hence all the complexes of bitangents are projections of double-sixes or else contain l.

The projections of the generators of a hyperboloid from any point O on to a plane all touch a conic, namely the conic in which the tangent-lines from O to the hyperboloid meet the plane. For the plane through O and a generator touches the hyperboloid.

Now the lines $a_1, a_2, a_3, b_4, b_5, b_6$ of a double-six lie on a hyperboloid, a_1, a_2, a_3 being generators of one family, and b_4, b_5, b_6, which meet them, being generators of the other family. Hence the corresponding bitangents of the quartic touch a conic. These lines can be chosen out of $a_1, a_2, \ldots, a_6$ in twenty ways. Hence:

In any complex of bitangents of a quartic there are twenty sets of six bitangents touching a conic. No two of these six bitangents belong to the same pair in the complex.

From the theorem that the intersections of each pair of a complex lie on a conic we obtain:

The six lines drawn through any point of a cubic surface each intersecting a pair of a given double-six lie on a cone of the second degree.

Ex. 1. By taking O of § 6 on a line of the cubic surface obtain properties of the bitangents of a quartic with a node.

Ex. 2. By taking O at the intersection of two lines of the surface obtain properties of a binodal quartic.

Ex. 3. The intersection of any plane with the tangent-lines from a point O to a cubic surface not passing through O is a sextic with six cusps. These cusps are the intersections of a conic with a cubic touching the sextic at the cusps; and the twenty-seven bitangents of the sextic are the projections from O of the twenty-seven lines on the surface.

CHAPTER XX

CIRCUITS

§ 1. Circuit and Branch.

HERETOFORE, when we have discussed singularities on a curve, we have not always distinguished real from unreal singularities. Such a statement as 'the number of inflexions of a non-singular n-ic is $3n(n-2)$' is only true if we include unreal inflexions. In the present chapter we shall make the distinction between real and unreal points on a curve, and by 'point', 'inflexion' mean 'real point', 'real inflexion', &c., unless the contrary is stated.

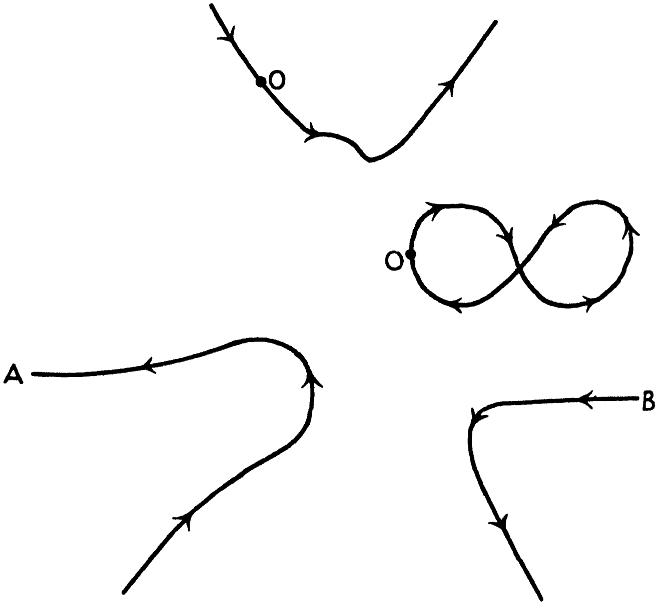


Fig. 1.

Suppose a point P starts from any point O on a curve, and travels along the curve, so that the direction of the tangent at P varies continuously. Suppose that, if in its journey P travels outwards to infinity from a very distant point A, it next travels inwards from infinity to a very distant point B such that A and B are neighbouring points on any projection of the curve. This means that in general P jumps from one end of an asymptote to the other. Eventually P returns to O. The part of the curve thus traced out by P is called a *circuit* of the curve. The portion of a circuit traced out by P between two infinitely distant positions of P is sometimes called a *branch* of the curve. Thus a hyperbola has one circuit composed of two branches.*

These definitions will apply to any continuous curve with continuously varying tangent. The curve need not be algebraic, though we shall be mainly concerned with algebraic curves.

Fig. 1 shows a curve with two circuits, one composed of three branches, and the other closed and crossing itself.

§ 2. Ovals.

Suppose that a circuit is closed, i. e. all its points are finite. Suppose, moreover, that it does not cross itself, i. e. has no crunode. Such a circuit will be called an *oval*. An oval can be reduced to a point by continuous deformation, as shown in Fig. 2 which gives successive positions of the oval as it shrinks to a point.

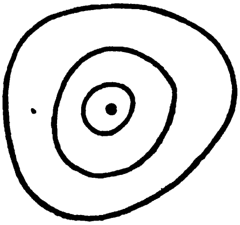


Fig. 2.

It is to be noticed that an acnode is not counted as a circuit. A curve loses one of its circuits, if an oval shrinks into a point.

§ 3. Odd and Even Circuits.

Suppose that a line is moving in the plane of a circuit. If the number of its intersections with the circuit alters, the number is increased or decreased by two (or some even number) as is obvious from Fig. 3, which shows three consecutive positions of the moving line. It follows at once that circuits can be divided into *even circuits* which are met by every straight line in an even number of points, and *odd circuits* which are met by every straight line in an odd number of points.

---

* In German, *Zug* ≡ circuit, *Ast* ≡ branch. Some English writers use 'part' for 'circuit', and speak of curves with one, two, three, ... circuits as 'unipartite', 'bipartite', 'tripartite', .. .

The reader must distinguish between the use of the word 'branch' as defined above and its use in such a phrase as 'branch of a curve at a multiple point'.

Any closed circuit, in particular any oval, is even; for it meets the line at infinity in an even (zero) number of points.

The oddness or evenness of a circuit is evidently unaltered by projection.

In general an n-ic has an odd or even number of odd circuits, as n is odd or even.

The least number of points in which a circuit is met by any line is called its *index*. The greatest number of points is called its *order*. The index and order are both odd or both even according as the circuit is odd or even.

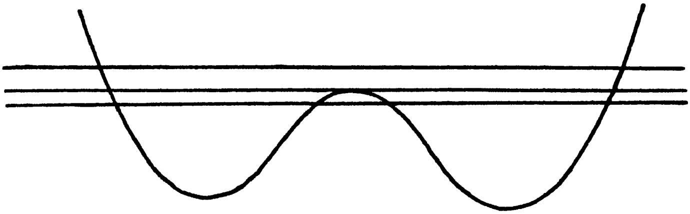


Fig. 3.

Two odd circuits meet in an odd number of points; two even circuits, or an odd and an even circuit, meet in an even number of points.

To prove this we first notice that, if one of the circuits is continuously deformed, and the number of its intersections with the other circuit alters, the number is increased or diminished by two (or other even number). In fact the argument of Fig. 3 still applies. We now show that the first circuit can be deformed so that all its points are very distant. Its intersections with the second (fixed) circuit will then be practically identical with its intersections with the asymptotes of the second circuit. But an odd number of lines meets an odd circuit in an odd number of points, an even number of lines meets an odd or even circuit in an even number of points, and any number of lines meets an even circuit in an even number of points. The theorem will therefore be established if we show how to deform the first circuit so that all its points are distant.

First of all deform the circuit so that its crunodes are 'cut' This is done by replacing the part near the node by the dotted line as shown in Fig. 4 (i). The method of Fig. 4 (ii) is inferior, for it would necessitate the alteration of the direction in which a moving point traces out the circuit, as shown by the arrow-heads.

The cutting of the nodes may really resolve the circuit into two or more distinct circuits, but for our present purpose we shall think of these distinct circuits as a single circuit whose

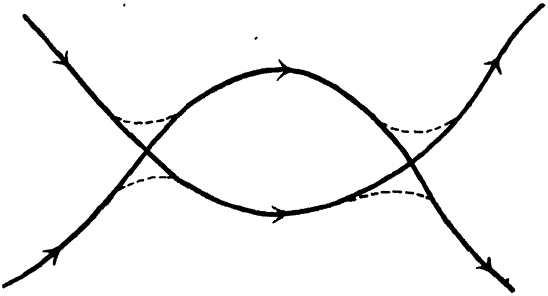


Fig. 4 (i).

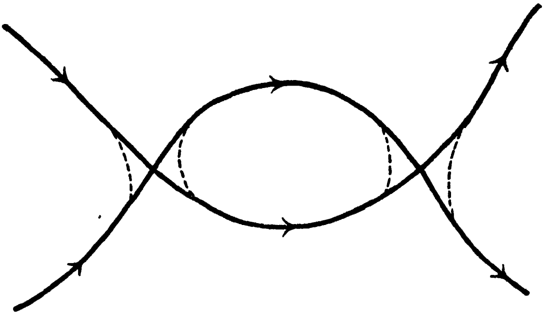


Fig. 4 (ii).

oddness or evenness is evidently unaltered by cutting the nodes.

The circuit now consists of distinct nodeless branches and ovals, no two of which cut. Keeping the two distant ends of each branch fixed, we can evidently withdraw such a branch continuously to an indefinitely great distance as shown in Fig. 5.

Also each oval can be deformed into a point; and we have achieved our purpose.

A nodeless curve cannot have more than one odd circuit.

For two odd circuits would meet in one or more nodes of the curve.

A non-singular n-ic has therefore one or no odd circuit as n is odd or even.

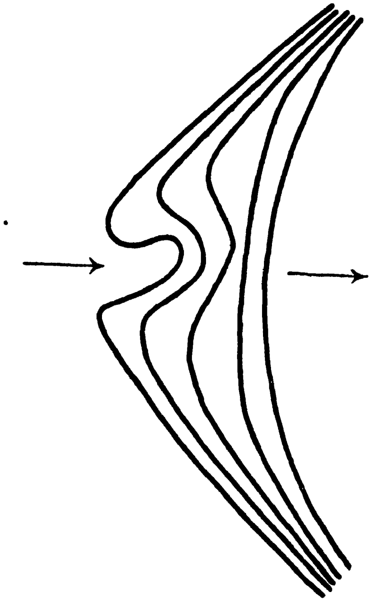


Fig. 5.

Ex. 1. An odd circuit is projected on to any sphere from its centre. Show that the projection is a closed circuit on the sphere met by any great circle in $4r+2$ points, whose points can be divided into diametrically opposite pairs.

Ex. 2. If an even circuit is projected similarly, the projection is a pair of diametrically opposite circuits each met by any great circle in an even number of points.

Ex. 3. Use Ex. 1, 2 to prove the theorem of § 3.

[Project after cutting the crunodes. On the sphere an even circuit is deformable into a point-pair and an odd circuit into a great circle in such a way that each pair of diametrically opposite points remains diametrically opposite.]

Ex. 4. The difference between the order and index of a circuit is even and greater than zero.

Ex. 5. An odd circuit of a (singular) quartic cannot meet any other circuit in more than one point. In particular, it cannot meet an even circuit.

[If it met in two, the line joining them would meet the quartic in five points.]

Ex. 6. Two odd circuits of a quintic cannot meet in more than one point.

A quintic with two intersecting even circuits cannot have more than one odd circuit.

Ex. 7. A closed circuit has n crunodes. Show that, if a node is cut as in Fig. 4 (i), the circuit is divided into two closed circuits. Also that, if all the nodes are cut, the number of circuits obtained differs from $n+1$ by an even number.

What happens if a node is cut the wrong way as in Fig. 4 (ii)?

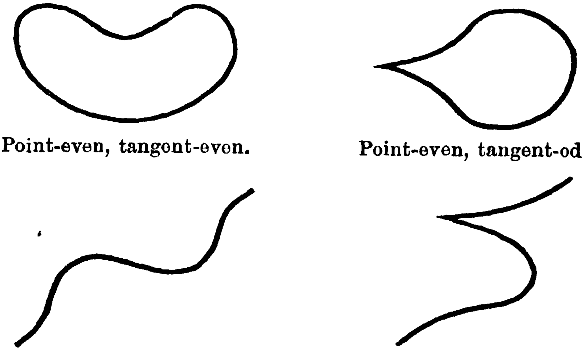
[See Landsberg, *Math. Annalen*, lxx (1911), p. 563.]

Ex. 8. Show that, if the circuit is even, but not closed, the cutting of a node either divides the circuit into two even circuits or into two odd circuits. In the latter case it is possible, by cutting a second node, to reconvert the figure into a single even circuit with two nodes less than the original.

[See Fig. 4.]

§ 4. Reciprocation of Circuits.

We call now an odd circuit 'point-odd' and an even circuit 'point-even'. A 'point-odd' circuit meets any line in an odd number of points. To its reciprocal with respect to any base conic we may therefore draw an odd number of tangents from any given point. The reciprocal will be called 'tangent-odd'. Similarly we obtain 'tangent-even' circuits as the reciprocals of point-even circuits. Fig. 6 shows the four possible varieties of circuit.



Point-even, tangent-even. Point-even, tangent-odd.

Point-odd, tangent-even. Point-odd, tangent-odd.

Fig. 6.

As a point P travels from a fixed point O on a circuit in a fixed direction, finally returning to O (§ 1), the tangent at P turns through an odd multiple of π, if the circuit is tangent-odd. For the tangent at P has passed an odd number of times through any fixed point A, since the number of tangents from A to the circuit is odd. Similarly the tangent at P turns through an even multiple of π, if the circuit is tangent-even.

§ 5. Inflexions and Cusps of Circuits.

A point-odd or point-even circuit contains respectively an odd or even number of inflexions. A tangent-odd or tangent-even circuit contains respectively an odd or even number of cusps.

One of these statements is the reciprocal of the other. We may confine our attention to the first.

Suppose that the tangent at a point P travelling along the circuit meets a fixed line l in Q. Let the tangent at a fixed point O of the circuit meet l in A (Fig. 7). As P starts from O,

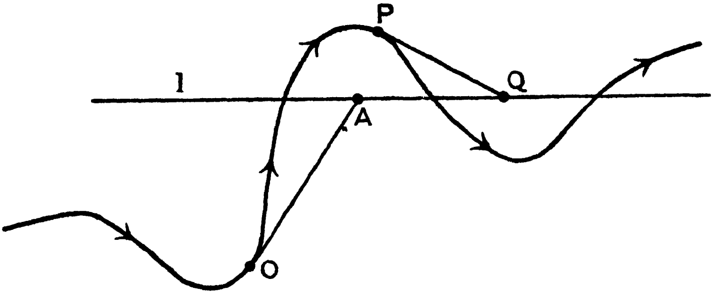


Fig. 7.

traverses the circuit, and returns to O, Q starts from A and finally returns to A, arriving at A in the same direction as it left A.

Hence $a+b+c$ must be even, a being the number of times Q passes through A,* b the number of times Q jumps from one end of l to the other, and c the number of times Q travels up to a certain point of l and then begins to move back again.

Now a is the number of tangents from A to the circuit, and b is the number of tangents parallel to l. Hence a and b are both odd or both even, according as the circuit is tangent-odd or tangent-even. Therefore $a+b$ is even; so that c is even.

But Q reaches a limiting position on l and then moves back again, if and only if P is an inflexion or an intersection of the circuit with l, which proves the result.

---

* The initial starting from A and the final arriving at A taken together count as passing through A *once*.

Ex. 1. If the tangent at P meets the circuit at Q_1, Q_2, Q_3, ..., when does the product $PQ_1 . PQ_2 . PQ_3 . \ldots$ change sign as P travels along the curve?

Deduce the theorem of § 5.

Ex. 2. An odd circuit without node or cusp has at least three inflexions.

[Project one inflexion to infinity.]

Ex. 3. A non-singular curve of odd degree has at least three real inflexions.

Ex. 4. An odd circuit of order three without node or cusp has exactly three inflexions (Mobius's theorem).

[Project one inflexional tangent to infinity. There is only one other tangent through the infinite inflexion, since the circuit is of order 3.

Let the tangent at P meet the circuit again at R. As P travels along the circuit, R travels along it twice in the opposite direction, and therefore coincides with P twice. Hence there are two finite inflexions.]

Ex. 5. Denoting the three parts into which the inflexions divide the circuit of Ex. 4 by l_1, l_2, l_3, show that there are two tangents from any point of l_1 whose points of contact lie on l_2 and l_3; and that the tangent from the inflexion at which l_2 and l_3 meet has its point of contact on l_1.

Show that if two lines meet the circuit in ABC, $A'B'C'$; A, A', B, B', C, C' cannot lie in this order on the circuit.

Ex. 6. An odd circuit has $2r+3$ inflexional tangents no one of which meets the circuit at more than three points (coinciding with the point of contact). Show that it has $r(2r+3)$ bitangents.

Show also that $2r+1$ tangents can be drawn from any inflexion, and $2r+2$ tangents from any other point of the circuit.

[Project one inflexion to infinity. The reader may enunciate a theorem concerning the positions of the points of contact similar to that of Ex. 5.]

Ex. 7. An even (odd) circuit has no node or cusp, but has $2p$ $(2p+3)$ inflexions and q bitangents. Show that $p-q$ is even.

[Deform the circuit into some standard position, and notice that during the deformation $p-q$ is never increased or diminished by an odd number.]

Ex. 8. The inverse of a circuit of r branches with respect to a point O not on the circuit is an even closed circuit with an r-ple point at O.

If O is an r-ple point of the circuit, the inverse circuit is odd or even as r is odd or even.

Ex. 9. An even number of circles of curvature of a circuit (i) pass through a given point O, (ii) cut orthogonally a given circle.

[(i) Invert with respect to O. (ii) Deform the circle continuously into its centre.]

Ex. 10. Describe the circuits of the curves in the diagrams contained in Ch. III to XIX.

[State whether point- or tangent-even or odd, give their order and index, &c.]

Ex. 11. Draw a one- or two-circuited cubic and show the portions of its plane from which 0, 2, 4, or 6 tangents can be drawn. Treat similarly some non-singular quartic curves.

[The curve and inflexional tangents divide up the plane into these various portions.]

Ex. 12. Every tangent to a circuit without node or bitangent meets it in the same number of points.

Ex. 13. Show that the result of Ex. 12 cannot be true for a circuit with a bitangent, but may be true for a circuit with a node.

[E.g. $x^2-y^2=x^2y^2$. Another example is

$$y^4-4y^2x^2+2y^3+x^2-2y^2=0.$$

The theorem is true for each circuit of this quartic, and for the quartic as a whole.]

Ex. 14 An oval without double point has at least four 'vertices', i.e. points at which the circle of curvature has four-point contact, and the radius of curvature is a maximum or minimum.

[Invert with respect to a point O of the oval and use Ex. 2, remembering that the circles of curvature at two points of the oval lying between two consecutive vertices cannot both pass through O.]

Ex. 15. If the oval of Ex. 14 has no inflexion, the sum of the maximum radii of curvature less the sum of the minimum radii of curvature is half the perimeter of its evolute.

§ 6. Method of Variation of Coefficients.

We shall require a method enabling us to obtain the equation of a curve with an assigned degree, number of circuits, inflexions, &c.

Such a method consists in giving small increments to the coefficients in the equation of some given curve C. The new equation thus obtained represents a curve of the same degree as C and lying very close to it. This is illustrated in Fig. 8, where the curve C is shown with a crunode, an acnode, a cusp, and two adjacent curves also.

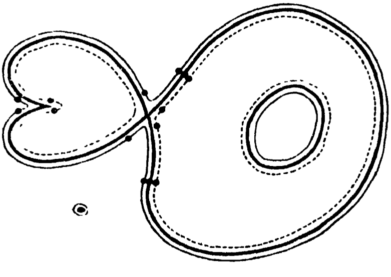


Fig. 8.

It will be noticed that in general, if C has a crunode while an adjacent curve has none, the adjacent curve has an inflexion close to each inflexion of C and two more inflexions close to the crunode of C. Similarly for a cusp.

In general also an adjacent curve on one side of C has a small oval enclosing the acnode of C, which is in general a convex oval without inflexions.*

The adjacent curve on the other side of C has no circuit near the acnode.

To each tangent from a node of C to C corresponds either two bitangents of an adjacent curve which are both real with real points of contact, or no real bitangent.

To each line joining two nodes of C corresponds four real bitangents of an adjacent curve having real points of contact, or no real bitangent.

The results for the case of a cusp are at once written down from Fig. 8.

We shall call a real bitangent with unreal points of contact an *ideal* bitangent. The name is due to W. K. Clifford. The number of ideal bitangents of an adjacent curve is the same as for C and these bitangents are adjacent to those of C, provided C has no imaginary nodes or cusps.

If, however, C has a pair of conjugate imaginary nodes or cusps E, F, an adjacent curve has respectively two or three ideal bitangents adjacent to EF.

For take the case in which E, F are nodes. Let Q be the quartic with a real cusp which approximates most closely to C at E, F; and as C passes into a consecutive position let Q pass into a consecutive quartic still approximating to C near E, F. The quartic in the position in which it has E, F as nodes consists of one circuit and has two real bitangents t_1, t_2, as may be readily proved by projecting E, F into the circular points and inverting with respect to the cusp. In the neighbouring position Q will have four bitangents of the first sort, i.e. bitangents whose points of contact are unreal or lie on the same circuit. This will be proved in § 8 by reasoning which does not involve the use of the result we are here establishing. Two of these bitangents are consecutive to t_1, t_2. The other two must be consecutive to EF. Now since the quartic Q approximates to the curve C near EF, a bitangent of C must approximate to each bitangent of Q near EF.†

A similar argument holds if E, F are cusps.

---

* Taking the acnode of C as origin, the adjacent curve has an equation of the form $0 = a + bx + cy + px^2 + 2qxy + ry^2 + sx^3 + \dots$, where $q^2 < pr$ and a, b, c are small. The approximation near the origin is the ellipse

$$0 = a + bx + cy + px^2 + 2qxy + ry^2.$$

† Zeuthen, *Math. Annalen*, vii, p. 424, reasons as follows: The line joining two nodes counts as four bitangents. If the nodes E, F are conjugate imaginary, the line EF counts as the limiting position of four bitangents. Two of these must be real bitangents in their final position just before becoming unreal, and the other two must be unreal becoming real, as C varies.

As illustrations of the above remarks which will be useful later on, let us take one or two theorems.

If a non-singular n_1-ic and n_2-ic exist having respectively $n_1(n_1-2)$ and $n_2(n_2-2)$ inflexions and no ideal bitangent, which meet in $n_1 n_2$ real points, there exists a non-singular (n_1+n_2)-ic with $(n_1+n_2)(n_1+n_2-2)$ inflexions and no ideal bitangent.

For if $f_1 = 0, f_2 = 0$ are the equations of the n_1-ic and n_2-ic, $f_1 f_2 = 0$ is an (n_1+n_2)-ic with $n_1 n_2$ crunodes,

$$n_1(n_1-2)+n_2(n_2-2)$$

inflexions, and no ideal bitangent. Then $f_1 f_2 = \epsilon$, where ϵ is any small constant, is a non-singular (n_1+n_2)-ic with

$$n_1(n_1-2)+n_2(n_2-2)+2n_1 n_2 = (n_1+n_2)(n_1+n_2-2)$$

inflexions and no ideal bitangent.

For any given value of n there exists a non-singular n-ic with $n(n-2)$ inflexions and no ideal bitangent.

Let us assume that such a curve exists and that moreover it meets an ellipse in the $2n$ (real) points. The reader will readily establish the truth of this statement in the cases $n = 2$ and $n = 3$.

Take $n+2$ lines each meeting the ellipse in real points. Then if $f = 0$, $e = 0$, $g = 0$ are the equations of n-ic, ellipse, and lines, $ef = \epsilon g$, where ϵ is a small constant, is an $(n+2)$-ic meeting the ellipse in the $2(n+2)$ real intersections of $e = 0$, $g = 0$. It has also $(n+2)n$ inflexions and no ideal bitangents as in the proof of the preceding theorem.

Now use induction.

Ex. 1. If $f = 0$ has k real branches through a point O, prove that one of the adjacent curves $f \pm \epsilon\phi = 0$ (where $\phi = 0$ does not go through O and ϵ is a small constant) has $2r$ inflexions near O and that the other has $2(k-r)$. If k is even, $r = \frac{1}{2}k$; but, if k is odd, r may have any of the values 0, 1, 2, ..., k.

Ex. 2 Obtain the equations of (i) a unicursal curve, (ii) a non-singular curve, of degree n consisting of one circuit only which is of index 0 or 1 as n is even or odd.

[(i) $x = f(t)$, $y = \phi(t)$, $z = \psi(t)$, where f, ϕ, ψ are polynomials of degree n and $\psi = 0$ has not more than one real root.

(ii) $(x^2+y^2-1)f = \epsilon$ or $(x-1)f = \epsilon$, where $f = 0$ is any unreal non-singular $(n-2)$-ic or $(n-1)$-ic with real equation.]

For we shall see in § 7 that the number of real bitangents of any non-singular n-ic adjacent to an n-ic with nodes at E and F is the same.

Perhaps neither line of argument will appear quite convincing. But the result is only required in the second part of § 7, and does not affect our discussion of the circuits of a non-singular curve in §§ 7, 8.

Ex. 3. A p-ic is cut by a straight line l in p real points, no two of which are consecutive. Show that an n-ic exists cutting the p-ic in np real points, n being any given integer.

[$f = \epsilon$, where $f = 0$ is the equation of n straight lines each meeting the curve in p real points. The line l can always be found if $p = 2$ or 3, but not if $p > 3$.]

§ 7. Klein's Theorem.

In any curve of degree n and class m

$$n + i + 2t = m + k + 2d,$$

where i is the number of real inflexions, t is the number of ideal bitangents (real, with unreal points of contact), k is the number of real cusps, and d is the number of real acnodes.

I. First consider the case of a non-singular n-ic. Suppose the coefficients in the equation of the curve to vary continuously. Then the curve is continuously changing its shape, and we may thus continuously deform the n-ic till it coincides with some standard non-singular n-ic.

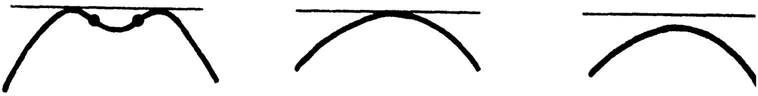


Fig. 9.
(The straight line is a bitangent in each case.)

We may suppose that during the deformation there is never more than one relation between the coefficients. During the deformation the curve may have one triple tangent, or one node, &c., but not a quadruple tangent, or flecnode, or pair of unreal nodes, &c.

Now in the process no two ideal bitangents can coincide, for this would imply the existence of a quadruple tangent.* Hence the only way in which $i + 2t$ can alter is

(i) by the coincidence of two inflexions at a point of undulation and their subsequent disappearance;

(ii) by the points of contact of a bitangent becoming real instead of unreal, or vice versa;

(iii) by the appearance of a node.

Now Fig. 9 shows that cases (i) and (ii) are really the same and that such an event decreases i by two and increases t by one, or vice versa. Also we have shown in § 6 that in case (iii) the numbers i and t are not permanently altered. For instance, if a crunode appears, i is diminished by two, and is increased by two again as the node disappears.

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* This would not apply to two bitangents with real points of contact, for they might coincide to form a triple tangent.

Hence $n+i+2t$ is the same for all non-singular n-ics.

But we have shown in § 6 that an n-ic exists for which

$$i=n(n-2), \quad t=0, \quad k=d=0, \quad m=n(n-1).$$

Hence the theorem is established for a non-singular curve.

II. Now suppose the n-ic has nodes and cusps. Let it have $\frac{1}{2}k'$ pairs of conjugate imaginary cusps, $\frac{1}{2}d'$ pairs of conjugate imaginary nodes, and d_1 crunodes, besides its k real cusps, d acnodes, i real inflexions, and t ideal bitangents.

An adjacent non-singular n-ic has by § 6 $i+2d_1+2k$ inflexions and $t+d'+\frac{3}{2}k'$ ideal bitangents. Therefore by Part I of the proof

$$n+(i+2d_1+2k)+2(t+d'+\tfrac{3}{2}k')=n(n-1).$$

Hence

$$\begin{aligned} n+i+2t &= n(n-1)-2(d_1+d+d')-3(k+k')+2d+k \\ &= n(n-1)-2\delta-3\kappa+2d+k=m+k+2d, \end{aligned}$$

as required.

The theorem of this section is due to F. Klein (*Math. Annalen*, x (1876), p. 199). It may be put in the form:

'The quantity $n+i+2t$ is the same for a curve and its polar reciprocal.'

As a corollary from Klein's theorem we deduce:

No n-ic has more than $n(n-2)$ real inflexions.

For a non-singular n-ic

$$i=m-n-2t=n(n-2)-2t$$

by Klein's theorem, which proves the result.

For an n-ic with double points it is sufficient to notice that we can find an adjacent non-singular n-ic for which i may be greater, but cannot be less.

We have already shown that for any assigned value of n an n-ic with $n(n-2)$ real inflexions exists.

Ex. 1. Klein takes as standard n-ic with $n(n-2)$ inflexions, &c., the following: For $n=2r$, a curve adjacent to the curve consisting of r equal concentric ellipses with major axes parallel to the r sides of a regular polygon. For $n=2r+3$, add to the ellipses a cubic whose odd circuit cuts all the ellipses in six points.

Verify this.

Ex. 2. What property of a curve do we obtain by applying Klein's theorem to its evolute?

[Use Ch. XI, § 2, Ex. 5, 6, 7.]

Ex. 3. Prove $\iota-3i+6d=\kappa-3k+6t$.

Ex. 4. A curve has no acnode and not more than a third of its cusps are real. Show that not more than a third of its inflexions are real.

[Use Ex. 3.]

Ex. 5. An n-ic with d_1 crunodes cannot have more than $n(n-2)-2d_1$ inflexions.

Ex. 6. What modification must be made in Klein's result, if the curve has a multiple point with distinct tangents?

[Apply Klein's theorem to an adjacent non-singular curve. Cf. § 6, Ex. 1. The reader may illustrate on $r = a \cos 3\theta$, which has the line at infinity as ideal bitangent.]

§ 8. Circuits of a Quartic.

We have already discussed (Ch. XIV, § 1) the possible circuits of a cubic. We consider here the case of the quartic, and discuss in detail the non-singular quartic.*

Such a quartic cannot have an odd circuit (§ 3). It cannot have more than four even circuits. For if it had five, the conic through a point on each circuit would meet the curve in ten points.

Klein's relation (§ 7) becomes $i + 2t = 8$ for the non-singular quartic. There is some line meeting the curve in no (real) point. If there is an inflexion, the reader will easily convince himself of the truth of this statement by projecting the inflexional tangent to infinity.† If there is no inflexion, there are four ideal bitangents each meeting the curve in no real point. The quartic can therefore be projected into a closed curve, which consists of one, two, three, or four ovals. Each oval is of order two or four.

If there are two ovals, they may lie one inside the other or be external to each other. In the former case the inner oval has no inflexion, for an inflexional tangent to the inner oval would meet the inner oval in four points and the outer oval in at least two, which is impossible for a curve of degree four.

If there are three or four ovals, no oval can lie inside another. For if a quartic had three ovals A, B, C with B inside A, a line cutting B and C would meet A, B, and C in at least two points each.

If two ovals are external to each other, it will be readily seen that they have four and only four common tangents which are bitangents of the quartic.* Such bitangents are called *bitangents of the second sort.*

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* See Ex. 6 for the case of quartics with double points.

† There is then a circuit approximating at infinity to a semi-cubical parabola ($ay^2 = x^3$) and with one asymptote also. This will be seen to have two branches with a common tangent, and a line adjacent to this tangent will meet the curve in no real point.

Bitangents of the first sort are those whose points of contact are either unreal or lie on the same oval.

A non-singular quartic has four bitangents of the first sort.

Imagine a string wrapped round an oval of order four. If AB is a straight portion of the string touching the oval at A and B, AB is a bitangent subtending a 'bay' ACB of the quartic with two inflexions (Fig. 10). Hence the quartic has twice as many inflexions as it has bitangents of the first sort with real points of contact.

The relation $i+2t = 8$ now proves the result.

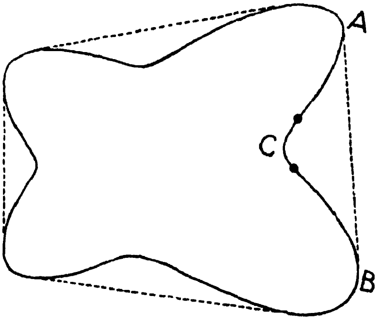

Fig. 10.

A non-singular quartic has 4, 8, 16, *or* 28 *real bitangents.*

It has four bitangents of the first sort. If it has r ovals external to each other, it has also $\frac{1}{2}r(r-1)\times 4$ bitangents of the second sort; where r is 1, 2, 3, or 4. Hence it has 0, 4, 12, or 24 bitangents of the second sort and four of the first sort, which proves the theorem.

Non-singular quartics may be classified by means of their ovals and the nature of their bitangents. We give an example to show how an equation may be found which shall represent a quartic belonging to an assigned class. Suppose we require a quartic with four ovals each with a bitangent. There will, of course, be also eight real inflexions and twenty-four bitangents of the second sort. The equation

$$(9x^2+4y^2-36)(4x^2+9y^2-36) = \epsilon,$$

---

* If the curve is of degree higher than five, there might be more than four such common tangents.

where ϵ is a small negative constant, will represent a quartic adjacent to the pair of ellipses

$$9x^2+4y^2=36, \quad 4x^2+9y^2=36,$$

and lying inside one ellipse and outside the other, as shown in Fig. 11 by the dotted line. It will evidently be a quartic of the kind stated.

If we take ϵ a small positive constant, we obtain a quartic with two ovals, one inside the other, as shown by the narrow line in Fig. 11. The outer oval has eight inflexions and four bitangents.

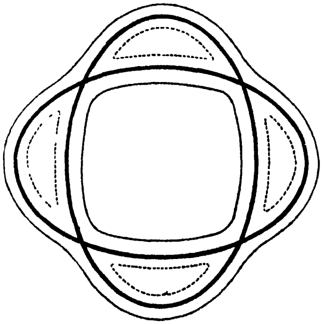

Fig. 11.

Ex. 1. The eight points of contact of the four bitangents of the first sort lie on a conic.

[Use Ch. XIX, § 2, Ex. 3. Each of the involutions is non-overlapping. See Ch. XIX, § 2, Ex. 6, for the case of four concurrent bitangents.]

Ex. 2. No triangle formed by three bitangents of the first sort can enclose two ovals external to one another.

[The argument of Ex. 1 would show that the conic through the points of contact of the three bitangents passes through the points of contact of one of the common tangents to the two ovals; which is impossible.]

Ex. 3. Describe the nature of the ovals and bitangents of the quartics $fg=\pm\epsilon$, where ϵ is a small positive constant, and $f=0$, $g=0$ are the curves given below:

(i) $x^2+y^2=1, \qquad 2x^2+y^2+1=0.$

(ii) $x^2+y^2\pm 4x+3=0.$

(iii) $x^2+y^2=4, \qquad 2x^2+y^2=1.$

(iv) $x^2+y^2\pm 4x-5=0.$

(v) $y^2+4y^2=4, \qquad 4x^2-y^2=4.$

(vi) $x^2-y^2=9, \qquad x^2+9y^2+2x=8.$

(vii) $x^2+4y^2=4, \qquad x^2-y^2=2x.$

Ex. 4. Describe the nature of the quartic $fg=\pm\epsilon\phi$, where

$$f\equiv x^2+4y^2-5, \qquad g=4x^2+y^2-5$$

and ϕ is

(i) $x+y-2$, (ii) $(x+y-2)(x-y+2)$,

(iii) $(x-1)(y-1)(x+y)$, (iv) $(x-y)(x+y-2)(x+y+2)$.

Ex. 5. Describe the nature of the quartic $(x^2+y^2-5y)^2=\pm\epsilon\phi$, where ϕ is

(i) $(y-1)(y-2)(y-3)(y-4)$, (ii) $(y-1)(y-2)(y-3)$,

(iii) $(y-1)(y-2)$, (iv) $(y-1)$,

(v) x^2-y^2, (vi) $(y-2x)(y-4)(y+x)$.

Ex. 6. Describe the nature of the circuits of the quartics in Ch. XVIII, §§ 3–15.

[The nature of the bicircular quartics may be obtained by inversion from the circular cubic. The nature of their bitangents is given by § 3, Ex. 13 and 16.

For § 4 take A and B at infinity and consider the possible positions of the conic of § 4, Ex. 1, relative to the tangents at A and B.

For §§ 5, 7, 14, 15 we may classify the various types of curve by considering the cases in which none, two, or four of a, b, c, d are unreal.

For § 9 project the curve into $x^2y^2+ax^2+by^2+c=0$, and similarly for §§ 11, 12.]

Ex. 7. To ovals of a quartic external to one another have two *external* bitangents (i.e. a bitangent such that the ovals lie on the same side of it) and two *internal* bitangents (the ovals lying on opposite sides).

Ex. 8. A quartic with three real cusps has one bitangent which is ideal. A conic meeting the curve in eight real points intersects the bitangent.

A quartic with one real and two unreal cusps has one bitangent with real points of contact. A conic meeting the curve in eight real points does or does not intersect the bitangent. In the latter case all the six common tangents of curve and conic are real.

[The quartic can be projected into a three-cusped hypocycloid or a cardioid.]

Ex. 9. Show that $x^3y+y^3z+z^3x=0$ has exactly six real inflexions and four real bitangents. The real inflexions lie on a conic.

[Drawing the curve, after putting $z=1$, we see that the quartic consists of a single oval touched in real points by three bitangents. The fourth bitangent is $x+y+z=0$.

The real inflexions are $(1, 0, 0)$, $(1+t, -\frac{1}{2}(1-t^2), 1-t)$, &c., where $t^3+1=t^2+9t$. They lie on $1/x+1/y+1/z=0$.]

Ex. 10. To two ovals of a quintic external to one another four common tangents can be drawn.

Ex. 11. No oval of a quintic can have more than twelve inflexions or six bitangents.

[Three real inflexions lie on the odd circuit, and no quintic has more than fifteen real inflexions.]

§ 9. Maximum Number of Circuits.

A curve of deficiency D has at most $D+1$ circuits.

This result, due to Harnack,* is familiar if $D=0$ (Ch. X, § 4). To prove it in general, suppose the curve to be of degree n with r odd circuits. These odd circuits meet in at least $\frac{1}{2}r(r-1)$ crunodes, since two odd circuits meet in one or more crunodes. Let the curve have s other double points, acnodes, unreal, or lying on odd or even circuits. Then

$$D=\tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}r(r-1)-s.$$

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* *Math. Annalen*, x (1876), p. 189.

Suppose that, if possible, the curve had $D+2$ or more circuits. It will have at least $D-r+2$ even circuits.

First consider the case in which n is even. Then r is even. Take one point on each of the $D-r+2$ even circuits, the $\frac{1}{2}r(r-1)+s$ double points, and $n+r-4$ more points on one of the even circuits. The total number of these points is

$$(D-r+2)+\tfrac{1}{2}r(r-1)+s+(n+r-4) = \tfrac{1}{2}(n+1)(n-2).$$

Hence a $(n-2)$-ic can be drawn through them. This $(n-2)$-ic meets each odd circuit of the n-ic at least twice; for it meets them each once at a crunode, and $n-2$ is even. Also it meets each even circuit in an even number of points, so that it meets one of them in at least $n+r-2$ points and the others at least twice. Hence the n-ic and $(n-2)$-ic meet in at least

$$2\{\tfrac{1}{2}r(r-1)+s\}+r+2(D-r+1)+(n+r-2) = n(n-2)+2$$

real points, which is impossible.

For the case of n and r odd, take one point on each of the even circuits, the $\frac{1}{2}r(r-1)+s$ double points, another point on each of the r odd circuits and $n-4$ more points on an even circuit.

Since $n-2$ is odd, the $(n-2)$-ic meets each odd circuit in three points at least, and the n-ic and $(n-2)$-ic meet in at least

$$2\{\tfrac{1}{2}r(r-1)+s\}+2r+2(D-r+1)+(n-3) = n(n-2)+1$$

real points, which is impossible.

Ex. 1. Show that not every non-singular n-ic can be projected into a closed curve, if n is a given number greater than 4.

[If n is odd, the curve can never be closed, for it has an odd circuit. For $n = 6$ take

$$(x^2-y^2-2)(x^2+8y^2+32y+28)(x^2+8y^2-32y+28) = \epsilon.]$$

Ex. 2. Let $\phi = 0$ be the equation of six lines parallel to the line of inflexions of a two-circuited cubic $f = 0$, four being on one side and two on the other side of the line of inflexions, and all meeting the odd circuit of the cubic in three points. Then $f^2 = \pm\epsilon\phi$ are (1) a sextic with twenty-four inflexions and eleven ovals, (2) a sextic with eighteen inflexions and nine ovals (i. e. projectable into such a sextic).

[To find all possible arrangements of the circuits of a non-singular n-ic is a difficult problem, which has been solved only for $n = 3, 4, 6$. For $n = 6$ we have the result: 'If a sextic has eleven ovals, they consist of ten ovals all external to each other and an oval enclosing either nine of them or only one'; as in the above example. The reader may discuss the cases in which $\phi = 0$ represent six or less lines in various positions.]

Ex. 3. Prove that

$$\{2r^3\sin 3(\theta+\delta)+3r^2-1\}\{8r^3\sin 3\theta+11r^2\}+\epsilon = 0,$$

where ϵ and δ are small and positive, is a sextic with eleven ovals.

§ 10. Nested Ovals.

A set of ovals $\omega_1, \omega_2, \ldots, \omega_r$ such that ω_1 lies inside ω_2, ω_2 inside $\omega_3, \ldots, \omega_{r-1}$ inside ω_r is called a family of r *nested ovals*.

If an n-ic has r nested ovals, we have:

(i) *n even;* $r \leqslant \frac{1}{2}(n-2)$, *or* $r = \frac{1}{2}n$, *the nest being the whole n-ic.*

(ii) *n odd;* $r \leqslant \frac{1}{2}(n-3)$, *or* $r = \frac{1}{2}(n-1)$, *the nest and one odd circuit being the whole n-ic.*

Suppose n even. Then, if the curve had $\frac{1}{2}n$ nested ovals and another circuit as well, a line cutting this other circuit

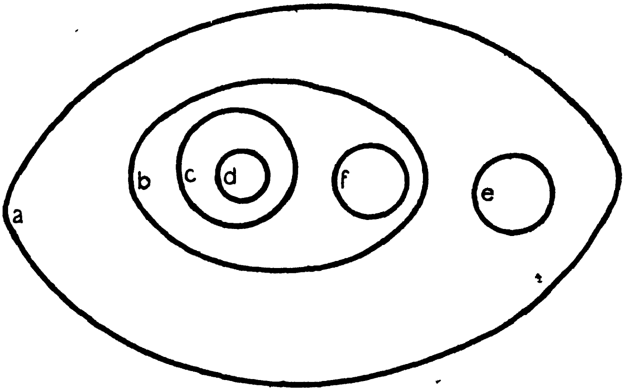

Fig. 12.

and the innermost oval would cut the n-ic in more than n points.

Suppose n odd. Then, if the curve had $\frac{1}{2}(n-1)$ nested ovals and also an even or two odd circuits in addition, a line cutting this even circuit (or two odd circuits) and the innermost oval would cut the n-ic in more than n points.

It will be noticed that a nest may contain ovals not belonging to the nest, as in Fig. 12. In this diagram we have the three nests *abcd*, *abf*, *ae*.

In the following section we prove the converse of this theorem, showing that for an assigned value of n it is always possible to find a non-singular n-ic with its maximum number of nested ovals and its maximum number of circuits and real inflexions.

Ex. Find the equation of an n-ic with $\frac{1}{2}n$ nested ovals when n is even, and $\frac{1}{2}(n-1)$ nested ovals and an odd circuit when n is odd.

$$[(x^2+y^2-1)(x^2+y^2-2)\dots(x^2+y^2-\tfrac{1}{2}n) = \epsilon x;$$

$$(x^2+y^2-1)(x^2+y^2-2)\dots(x^2+y^2-\tfrac{1}{2}\{n-1\})(x-n) = \epsilon.]$$

§ 11. Hilbert's Theorem.

For every assigned value of n greater than 3 there exists an n-ic with the following properties:

(i) *It is non-singular.*

(ii) *It has the maximum number $\frac{1}{2}n(n^2-3n+4)$ of circuits.*

(iii) *It has the maximum number $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ of nested ovals.*

(iv) *It has the maximum number $n(n-2)$ of real inflexions.*

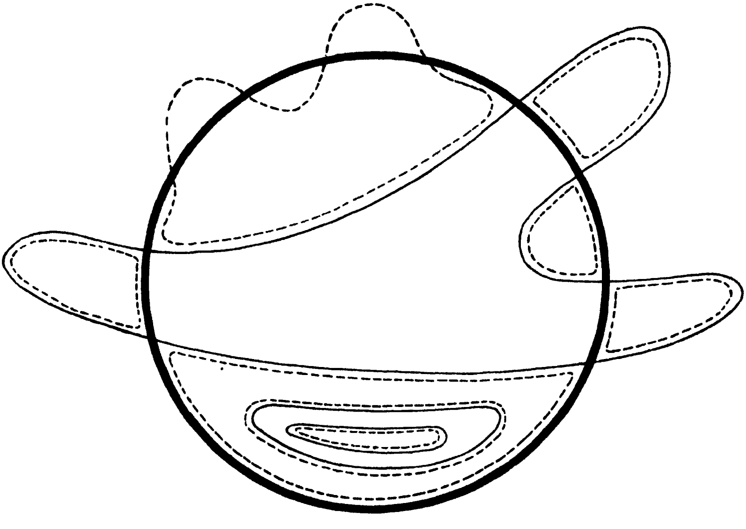

Fig. 13.

The theorem may be considered to hold even in the cases $n = 2$ or 3, if we adopt the convention that a single oval shall count as a nest of one oval when $n > 3$ and shall not count as a nest when $n = 2$ or 3.

Suppose that in Fig. 13 we have a circle (shown by the thick line) with equation $e = 0$. Suppose we have also a curve (shown by the thin line) with equation $f = 0$, which possesses a nest of p ovals, such as is shown at the bottom of the diagram, and an oval Ω meeting the circle in $2q$ points (only) passed through in the same order whether we traverse the circle or the oval Ω. There is a portion of the plane not containing the nest, which is bounded by an arc of the circle, and an arc of Ω lying inside the circle. On this arc of the circle take $2r$ points $B_1, B_2, \ldots, B_{2r}$; and let the equation of the lines $B_1B_2, B_3B_4, \ldots, B_{2r-1}B_{2r}$ be $g = 0$.

In the diagram $p = 2$, $q = 3$, $r = 2$.

Then, if ϵ is a small constant of suitable sign.

$$ef = \epsilon g$$

is a curve such as is shown by the broken line in Fig. 13.

It has a nest of $p+1$ ovals and has in all $2q-1$ more ovals than the original curve. It has too $4q$ more inflexions, since each intersection of the circle and original curve gives rise to two extra inflexions by § 6.*

It has also an oval of the same nature as the oval Ω, meeting the circle in $2r$ points passed through in the same order whether we traverse the circle or the oval.

Supposing now we have an n-ic $f = 0$ satisfying the conditions of the theorem, with $\frac{1}{2}(n^2-3n+4)$ circuits of which $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ form a nest as in Fig. 13 and one is an oval such as Ω meeting a fixed circle $e = 0$ in $2n$ points ($q = n$). Suppose also the n-ic has $n(n-2)$ real inflexions. If we take $r = n+2$, the derived curve

$$ef = \epsilon g$$

is an $(n+2)$-ic with a nest of $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ ovals and an oval such as Ω meeting the circle in $2(n+2)$ points. The derived curve has also

$$n(n-2)+4n = (n+2)n$$

real inflexions, and the number of its circuits is

$$\tfrac{1}{2}(n^2-3n+4)+2n-1 = \tfrac{1}{2}\{(n+2)^2-3(n+2)+4\}.$$

We have now only to establish the existence of the n-ic with the nest, the oval Ω, and the $n(n-2)$ inflexions for the cases $n = 2$ and $n = 3$.

The result will then follow by induction.

For $n = 2$ it is sufficient to take any ellipse meeting the circle in four real points. For $n = 3$ we may take the curve

$$eh = \epsilon g,$$

where $e = 0$ is a circle, $h = 0$ a straight line not meeting the circle, and $g = 0$ three straight lines all meeting the circle.

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* Fig. 13 is purely diagrammatic, and inflexions are shown in the figure which do not really exist.

The reader who wishes to pursue further the subject of circuits may consult:

American Journal Math., xiv, p. 245; xxix, p. 305.

Annali di Mat. Pura ed Applicata, III, xxii, p. 117.

Berichte der K. Sächsischen Gesell. der Wiss. zu Leipzig, lxiii, p. 540.

Bull. New York Math. Soc., i, p. 197.

Crelle, cxiv, p. 170.

Math. Annalen, vii, p. 410; x, p. 189; xxxviii, p. 115; xli, p. 349; lxvii, p. 126; lxix, p. 218; lxxiii, p. 177; lxxiv, p. 319; lxxvii, p. 416.

Rendiconti del Reale Istituto Lombardo, II, xliii (1910), pp. 48 and 143; xlvii (1914), pp. 489 and 797; xlviii (1915), p. 182; xlix (1916), pp. 495 and 577.

Trans. American Math. Soc., iii, p. 388.

Ex. 1. A curve $f=0$ has a circuit meeting a line $e=0$ in q points (only) which are passed through in the same order whether we traverse the line or the circuit. The r lines $g=0$ meet $e=0$ in r points, such that the two segments containing the q points and the r points on $e=0$ do not overlap. Prove that $ef=\epsilon g$, where ϵ is a small constant of suitable sign, has $q-1$ more circuits than $f=0$, and has a circuit meeting $e=0$ in r points only which are passed through in the same order whether we traverse the line or the circuit.

Ex. 2. A non-singular n-ic exists for any given value of n, having its maximum number of circuits, with a circuit meeting a given line in n points passed through in the same order whether we traverse the line or the circuit.

[Take $q=n$, $r=n+1$ in Ex. 1, and use induction.]

Ex. 3. $(x^2+y^2)(y-ax)(y-cx)\dots(y-kx)=y(y-bx)(y-dx)\dots(y-lx)$, where $a, b, c, d, \dots, k, l$ are $2n-4$ constants in ascending order of magnitude, is an n-ic of zero deficiency with a single circuit of index $n-2$.

CHAPTER XXI

CORRESPONDING RANGES AND PENCILS

§ 1. Correspondence of Two Pencils.

SUPPOSE we take two lines PA, PB $y = tz$, $x = Tz$ respectively through the points A and B of the triangle of reference ABC, such that t and T are connected by a relation of the form

$$\begin{aligned} T^q(a_0t^p + a_1t^{p-1} + \dots + a_p) + T^{q-1}(b_0t^p + b_1t^{p-1} + \dots + b_p) \\ + \dots + (k_0t^p + k_1t^{p-1} + \dots + k_p) = 0 \quad . \quad . \quad \text{(i)}. \end{aligned}$$

To every position of PB correspond p positions of PA, and to every position of PA correspond q positions of PB. The lines PA and PB are said to trace out *pencils with vertices A and B having a $p:q$ correspondence.*

Eliminating t and T between (i) and $y = tz$, $x = Tz$ we obtain the locus of P. It is a $(p+q)$-ic with multiple points at A and B of orders p and q respectively. The p tangents at A are the p lines of the pencil with vertex A which correspond to the line BA with vertex B; and similarly for the tangents at B. These facts are obvious either from the equation of the curve or from simple geometrical considerations.

Plücker's numbers are at once written down, remembering that a k-ple point counts as $\frac{1}{2}k(k-1)$ nodes (Ch. VIII, § 3). We have *

$$\left.\begin{aligned} &n = p+q, \quad m = 2pq, \quad \delta = \tfrac{1}{2}(p^2+q^2-p-q), \quad \kappa = 0, \\ &\tau = 2(p^2q^2-5pq+2p+2q), \qquad \iota = 3(2pq-p-q), \\ &D = (p-1)(q-1) \end{aligned}\right\} .$$

The tangents from A to the curve are the p tangents at A each counted twice and the $2(q-1)p$ lines given by those values of t which make (i), considered as an equation in T, have equal roots. We verify easily from this that $m = 2pq$.

In (i) we have supposed $a_0 \neq 0$. If $a_0 = 0$, the line AB corresponds to itself in the two pencils. The locus of P is now the line AB together with a $(p+q-1)$-ic having a $(p-1)$-ple point at A and a $(q-1)$-ple point at B. For this curve

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* Assuming that the tangents at A and B are all distinct, and that the curve has no multiple point other than A and B. This assumption may not be valid, if certain relations hold between the coefficients of equation (i).

$$\left.\begin{array}{l} n = p+q-1, \quad m = 2(pq-1), \quad \delta = \frac{1}{2}(p^2+q^2-3p-3q+4), \\ \kappa = 0, \quad \tau = 2(p^2q^2-7pq+2p+2q+4), \\ \iota = 3(2pq-p-q-1), \quad D = (p-1)(q-1) \end{array}\right\}$$

The case $p = q = 1$ is very well known. We have then that the intersection of corresponding rays of two homographic pencils is a conic, or is a straight line if the line joining the vertices of the pencils corresponds to itself. The reader may also verify the results obtained by taking $p = 2$, $q = 1$ or $p = q = 2$.

If $a_0 = a_1 = b_0 = 0$, the locus of P is similarly a $(p+q-2)$-ic with $(p-2)$-ple and $(q-2)$-ple points at A and B respectively; and so on.

We readily show that, conversely, the lines PA and PB joining any point P on a $(p+q)$-ic to multiple points A and B of orders p and q respectively have a $p:q$ correspondence. In fact, putting $y = tz$ and $x = Tz$ in the equation of the curve we have a relation of the form (i).

More generally, any n-ic with an r-ple point A and an s-ple point B* may be considered as the locus of P, when the pencils traced out by PA and PB have a $(n-s):(n-r)$ correspondence such that $n-r-s$ of the lines through A corresponding to BA coincide with AB, and $n-r-s$ of the lines through B corresponding to AB coincide with BA.

This is evident on putting t for y/z, T for x/z in the equation of the curve, when we get a relation of the form (i) with

$$a_0, a_1, \ldots, a_{n-r-s-1}; \; b_0, b_1, \ldots, b_{n-r-s-2}; \; \ldots$$

all zero.

Ex. The quadratic transform of the $(p+q)$-ic of § 1 with respect to a conic touching CA and CB at A and B is the intersection of pencils through A and B having a $q:p$ correspondence.

[The quadratic transform of a line through A is a line through B.]

§ 2. Correspondence of Two Ranges.

The polar reciprocals of two pencils with a $p:q$ correspondence are ranges of points on two lines with a $p:q$ correspondence. To any point of the first range correspond q points of the second, and to any point of the second range correspond p points of the first. We have a relation such as (i) of § 1, if t and T are the distances of corresponding points measured from fixed origins on the two lines.

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* $r = 0$ if A is not on the curve, $r = 1$ if A is an ordinary point of the curve; and so for B.

Two pencils with a $p:q$ correspondence meet any two lines in ranges with a $p:q$ correspondence, and the pencils formed by joining any two vertices respectively to the points of ranges with a $p:q$ correspondence have themselves a $p:q$ correspondence.

The line joining corresponding points of two ranges with a $p:q$ correspondence in general envelops a curve of class $p+q$ having the lines on which the ranges lie as p-ple and q-ple tangents respectively.

Ex. 1. Two ranges on the same line with $p:q$ correspondence have $p+q$ self corresponding points.

Two pencils with a common vertex and $p:q$ correspondence have $p+q$ self-corresponding rays.

Ex. 2. A, B, C are fixed points. Any line through A meets a fixed conic through B in P and a fixed line in Q. Find the locus of the intersection of BP and CQ.

[The pencils CQ and BP have a 1 : 2 correspondence. The locus is therefore a cubic through C with a node at B.]

Ex. 3. Find the locus of the intersection of tangents from two fixed points A, B to a given family of confocal conics.

[The tangents from A and B have a 2 : 2 correspondence in which AB is self-corresponding. The locus is a cubic through A and B.]

Ex. 4. Find the locus of the foci of curves of the m-th class touching $m-1$ given lines when the line at infinity (i) is not, (ii) is, one of the given lines.

[(i) A $(2m-1)$-ic with $(m-1)$-ple points at the circular points. The singular foci of the locus lie on the locus.

(ii) A $(2m-2)$-ic with $(m-1)$-ple points at the circular points.

Consider the correspondence of the tangents to the curve from the circular points.

See also Ch. V, § 2, Ex. 5.]

Ex. 5. A, B, C are fixed points. Through A and C are drawn conjugate chords of a fixed conic through A and B meeting the conic in P and Q respectively. Find the locus of the intersection of AP and BQ.

[The pencils AP and BQ have a 1 : 2 correspondence.]

Ex. 6. A conic is drawn through three fixed points A, B, C to touch a fixed line l at any point P. The tangent at A meets l at Q. Find the locus of the intersection of BQ and CP.

[The pencils BQ and CP have a 1 : 2 correspondence.]

Ex. 7. A conic is drawn through four fixed points A, B, C, D and meets a fixed line through A in P and another fixed line in Q. Find the locus of the intersection of BP and CQ.

[The pencils BP and CQ have a 1 : 2 correspondence, BC being self-corresponding.]

Ex. 8. A, B, C, D, E, F are fixed points. Any conic through A, B, C, D meets two fixed lines in P and Q respectively. Find the locus of the intersection of EP and FQ.

[The pencils EP and FQ have a $2:2$ correspondence.]

Ex. 9. A, B, C, D are fixed points Tangents are drawn from A and B to any conic of a given confocal family to meet a given line in P and Q respectively. Find the locus of the intersection of CP and DQ.

[The pencils CP and DQ have a $2:2$ correspondence.]

Ex. 10. A, B, C, D, E are fixed points. A fixed conic passes through C. A variable conic touches fixed lines at A and B, and meets the tangent at C to the fixed conic in P. A common tangent of the two conics meets this tangent at C in Q. Find the locus of the intersection of DP and EQ.

[The pencils DP and EQ have a $2:4$ correspondence.]

Ex. 11. A, B, C, D, E, F, G, H, I are fixed points. Any cubic through A, B, C, D, E, F, G, H meets a fixed line at P. Find the locus of the intersection of the tangent at A and the line IP.

[The tangent and IP have a $1:3$ correspondence.]

Ex. 12. How many solutions are there to the problem: 'Draw a circle through two given points to meet two given lines in points P, Q collinear with a given point O'?

[OP and OQ have a $2:2$ correspondence. Therefore by Ex. 1 the answer is $2+2=4$.]

Ex. 13. How many circles can be drawn through a fixed point A touching a given line at P such that P and the intersection Q of another fixed line with the tangent at A are collinear with a fixed point O?

[OQ and OP have a $1:2$ correspondence. Therefore the answer is $1+2=3$.]

Ex. 14. How many solutions are there to the problem: 'Draw a circle through two fixed points A, B meeting a given conic at P and a given line at Q so that PQ meets the conic again at a given point O'?

[OP and OQ have a $4:2$ correspondence. The answer is 6. Discuss the case in which the given line passes through A.]

Ex. 15. If x is the distance of a point on a given line from a fixed origin and $f(x)$, $\phi(x)$ are polynomials of degree n, the group of n points given by $f(x)=k\phi(x)$ is said to trace out an *involution-range of degree n* as k varies. A similar definition holds for pencils. Show that the involution-range has $2(n-1)$ double points.

Ex. 16. The two involutions of degree n and N given by the equations

$$f(x)=k\phi(x) \qquad \text{and} \qquad F(y)=k'\Phi(y),$$

are said to be *projective*, if k and k' are connected by a relation of the form $akk'+bk+ck'+d=0$. Show that two projective involution-ranges on the same line have $n+N$ self-corresponding points; and that the line joining corresponding points of two projective involutions along different lines envelops a curve of class $n+N$ having the lines containing the involutions as n-ple and N-ple tangents respectively. Show also that a corresponding result holds for pencils.

Ex. 17. Two families of curves have respectively degree, class, characteristic n_1, m_1, (p_1, l_1) and n_2, m_2, (p_2, l_2). Show that, if the families have a 1 : 1 correspondence, the locus of the intersection of corresponding curves is in general of degree $n_1p_2+n_2p_1$ and of class $m_1l_2+m_2l_1$. (See Ch. IV, § 8.)

[Corresponding curves have equations

$$f_1(x, y, k) = 0 \quad \text{and} \quad f_2(x, y, k) = 0,$$

where f_1 and f_2 are of degrees n_1 and n_2 in x and y, p_1 and p_2 in k. The intersections of the locus with $y = 0$ are found by eliminating k from

$$f_1(x, 0, k) = 0 \quad \text{and} \quad f_2(x, 0, k) = 0.]$$

§ 3. Curves with a One-to-One Correspondence.

We may use our knowledge of the locus of the intersection of corresponding rays of two pencils to establish the theorem:

If the points of two curves have a 1 : 1 *correspondence, the curves have the same deficiency.*

This means that, if to each point P with Cartesian coordinates (x, y) on one curve corresponds a point P' with coordinates (x', y') on the other so that x', y' may be expressed rationally in terms of x, y and vice versa, then the curves have the same deficiency.

It is at once seen that, if points P, P', P'' are taken on three curves, such that the coordinates of P can be expressed rationally in terms of P' and vice versa, while the coordinates of P' can be expressed rationally in terms of P'' and vice versa, then the coordinates of P can be expressed rationally in terms of P'' and vice versa.

Now we have proved (Ch. IX, §§ 1, 8) that two curves derived from each other by quadratic transformation have a 1 : 1 correspondence and have also the same deficiency. It will suffice therefore to prove the result for any two curves into which the given curves may be transformed by a series of quadratic transformations. Now by a series of quadratic transformations we may transform a given curve successively into curves with less and less complex singularities, till we obtain a curve with no singularities other than ordinary multiple points with distinct tangents. Hence it suffices to prove the theorem for two curves with ordinary multiple points with distinct tangents only.

Suppose Q, R corresponding points on two such curves of degrees n, N and classes m, M. Take two fixed vertices A, B, and let AQ, BR meet in P. The locus of P is an algebraic curve; for we may eliminate the coordinates of Q and R from the rational equations connecting them, from the equations of the given curves, and from the equations of the lines BR and AQ, thus getting an algebraic eliminant which is the equation of the locus of P. To each position of AQ correspond n positions of BR, and to each position of BR correspond N positions of AQ. Hence AQ and BR have an $N:n$ correspondence, and the locus of P is an $(n+N)$-ic, with A and B as N-ple and n-ple points.

The tangents from A to the locus of P are the N tangents at A each counted twice, together with the m lines through A touching the locus of Q. For two of the lines BR corresponding to such a position of AQ coincide. Hence the class of the locus of P is $2N+m$. Similarly it is $2n+M$. Therefore $m-2n+2$ and $M-2N+2$ are equal. But these are twice the deficiencies of the curves since they have no cusps (Ch. VIII, § 3, Ex. 1).

For example, we pointed out that the coordinates (ξ, η) of a point on the evolute corresponding to a point (x, y) of a given curve $f(x, y) = 0$ are expressible rationally in terms of x and y by means of equations (i) of Ch. XI, § 2.

Conversely, when we solve for x in terms of ξ, η by eliminating y from these equations, making use of $f(x, y) = 0$, we express x rationally in terms of ξ, η. For otherwise (ξ, η) would be the centre of curvature at more than one point of $f(x, y) = 0$, which is not in general the case. Similarly for y.

Hence a curve and its evolute have a 1 : 1 correspondence, and have therefore the same deficiency.

Ex. 1. A curve and its reciprocal have the same deficiency.

[See Ch. VIII, § 1 (vi).]

Ex. 2. The Hessian, Steinerian, and Cayleyan of a curve have the same deficiency.

Ex. 3. If p and q are relatively prime integers, p being positive, the deficiency of

$$x^{\frac{p}{q}} + y^{\frac{p}{q}} + 1 = 0 \quad \text{is} \quad \tfrac{1}{2}(p-1)(p-2).$$

[The transformation $x = \xi^q$, $y = \eta^q$ establishes a 1 : 1 correspondence between the given curve and $x^p + y^p + 1 = 0$ which is non-singular of degree p. See V. Jamet, *Bull. de la Soc. Math. de France*, xvi (1888), p. 132.]

§ 4. Correspondence in Three Dimensions.

Just as in § 1 we had two pencils of lines with $p:q$ correspondence, so we may have two pencils of planes with $p:q$ correspondence, the planes of each pencil passing through a given line called the *axis* of the pencil.

Corresponding planes meet on a ruled surface of degree $p+q$ having the axes of the pencil as p-ple and q-ple lines respectively. This is evident from the fact that any plane meets the two pencils of planes in two pencils of lines with a $p:q$ correspondence; or it may be proved independently as in § 1.

If the axes of the pencil intersect at O, the ruled surface becomes a cone with vertex O.

Reciprocating, the line joining corresponding points of two ranges on non-intersecting lines with a $p:q$ correspondence generates a ruled surface Σ such that $p+q$ tangent planes can be drawn to Σ through any line l. But these tangent planes are the planes joining l to the generators through the intersections of l with Σ; so that these intersections are $p+q$ in number. Moreover, the reciprocation shows that the lines on which the ranges lie are q-ple and p-ple lines of Σ.

We see then that the line joining corresponding points of two ranges on non-intersecting lines with a $p:q$ correspondence generates a ruled surface of degree $p+q$ with the given lines as q-ple and p-ple lines respectively.

The case $p = q = 1$ is well known.

Ex. If the coordinates of two points P and P', one on each of two given twisted curves, are connected by rational relations so that p points P correspond to each position of P' and q points P' to each position of P, then the line PP' traces out an algebraic ruled surface of degree $p+q$.

[The planes joining any given line l to P and P' have a $p:q$ correspondence, and the $p+q$ self-corresponding planes give the generators of the surface which meet l.]

§ 5. Curves on a Conicoid.

The $(p+q)$-ic of § 1 may be employed to study algebraic curves on a conicoid, that is, the whole or partial intersection of the conicoid with any algebraic surface.*

Let O be a point of the conicoid j, and let any plane Π meet the generators through O in A and B.

Consider the projection from O on to Π of any curve on j, which does not pass through O. All generators of one family on j intersect OA; they therefore project into straight lines through A. Similarly the other family of generators project into straight lines through B.

Suppose that the curve on j we are investigating is the intersection of j with a surface of degree n having a generator of the same family as OA as an $(n-q)$-ple line, and a generator of the same family as OB as an $(n-p)$-ple line, these two generators not being counted as part of the curve of intersection. Then OA meets the curve in p points all projecting into the point A, and OB meets the curve in q points all projecting into B. Any generator of the same family as OA

---

* Every twisted curve of the third or fourth degree is of this nature; for a conicoid through nine points of the curve must contain the curve.

meets the curve in p points, and projects into a line meeting the projection of the curve q times at B and p times elsewhere. Similarly for a generator of the other family. Hence the projection of the curve is a $(p+q)$-ic with A as p-ple point and B as q-ple point The tangents at A to the projection lie in the planes through OA touching the twisted curve where it meets OA; and so for B.

Each inflexion of the plane curve is the projection of a point of the twisted curve at which the osculating plane passes through O.

Every property of the plane curve gives a property of the twisted curve on projecting back on to the conicoid j; and vice versa.

A conic on j projects into a conic through A and B; and conversely. This is clear on putting $p = q = 1$, or otherwise. Let V be the pole of the plane of a conic on j, which meets OA and OB in P and Q. Then OVP and OVQ are the tangent planes at P and Q to j. Hence the lines VP and VQ project into the tangents at A and B to the projected conic; i.e. the pole of the plane of a conic on j projects into the pole of AB with respect to the projected conic.

If the plane of any other conic on j passes through V, the polar of V with respect to it is the line of intersection of its plane with the polar plane of V. Projecting we see that, if the planes of two conics on j are conjugate, their projections are two conics through A and B such that the pole of AB for one conic is the pole of their other common chord for the second conic; in other words, the projections are two conics through A and B with degenerate harmonic locus and envelope.*

We have so far supposed that the point O does not lie on the given twisted curve. If O does lie on this curve, the generators OA and OB meet the curve respectively in $p-1$ and $q-1$ points other than O. If we project on to the plane Π, we obtain a $(p+q-1)$-ic with A as $(p-1)$-ple point and B as $(q-1)$-ple point. The projection meets AB again at a point P on the tangent at O to the given twisted curve, and the tangent at P to the projection lies in the osculating plane of the curve at O.

A twisted curve on j meeting all generators of one family in p points and all generators of the other family in q points may be called a curve *of type* $p+q$ on the conicoid. The curves of type $2+1$, $2+2$, $3+1$ are well known.

---

* This is obvious on projecting A and B to the circular points, when the conics become orthogonal circles. The harmonic envelope is the envelope of a line divided harmonically by the conics, and the harmonic locus is the locus of a point from which a harmonic pencil of tangents can be drawn to the conics.

Any twisted cubic curve is the partial intersection of two conicoids with a common generator and is of the type $2+1$. For since a twisted cubic meets any conicoid not containing it in six points, it must lie wholly on all conicoids through any eight points of the curve.

Similarly a twisted quartic lies on the conicoid through any nine points of the curve. If it lies on a second conicoid, it is of type $2+2$; otherwise of type $3+1$.

Ex. 1. The points of contact of the three osculating planes of a twisted cubic which pass through any point O lie on a plane through O.

[The cubic lies on a conicoid through O. Projecting from O we have: 'The three inflexions of a plane nodal cubic are collinear.']

Ex. 2. With any point O as vertex three cones of the second degree can be drawn having six-point contact with a given twisted cubic. The tangent planes to the cone at the points of contact pass through the points of contact of the osculating planes through O.

Ex. 3. Four tangent lines of a given twisted cubic meet any given line in space.

[Project from a point on the line.]

Ex. 4. Through the line joining a fixed point O to any point of a twisted cubic two tangent planes are drawn to the curve touching at Q and R. Show that the plane OQR envelops a cone of the second degree.

[See Ch. XIII, § 4, Ex. 13 (iii).]

Ex. 5. The planes joining four given points P, Q, R, S of a twisted cubic to any chord form a pencil of constant cross-ratio.

[Let AB and CD be two chords. Projecting from A, we see that the pencils $AB(PQRS)$ and $AC(PQRS)$ have the same cross-ratio. Similarly project from C for the pencils joining CA and CD to P, Q, R, S.]

Ex. 6. If in Ex. 5 the pencil is harmonic, the tangent at P meets the osculating plane at R in a point on the plane RQS.

[Project from R.]

Ex. 7. Nine osculating planes of a curve of type $2+2$ pass through a given point O on the curve. The plane through O and any two of the points of contact passes through a third point of contact.

[Project from O.]

Ex. 8. Through any variable chord of a curve of type $2+2$ four tangent planes can be drawn, and the cross-ratio of the pencil of planes is constant.

[Let AB and CD be two chords. Projecting from A, we have the fact that the pencils of tangent planes through AB and AC have the same cross-ratio. Similarly project from C for the chords CA and CD.]

Ex. 9. The tangent planes through any point O to a curve of type $2+2$ at the points where it meets the generators through O of a conicoid containing the curve meet the curve again in four coplanar points.

[Projecting from O we have Ch. XVIII, § 4, Ex. 1.]

Ex. 10. Obtain properties of a twisted curve of type $3+1$ by projecting from any point of the conicoid containing the curve.

[See Ch. XVII, § 8.]

Ex. 11. The three points of contact of the osculating planes of a curve of type $3+1$ which pass through a point O of the curve lie on a plane through O.

Ex. 12. O is a fixed point on a curve of type $3+1$, and P is any other point on the curve. The tangent planes through OP to the curve touch at Q and R. Show that the plane OQR envelops a cone of the second degree.

Ex. 13. The line of striction of one family of generators of a hyperboloid is a curve of type $3+1$.

§ 6. Curves on a Sphere.

If the conicoid j of § 5 is a sphere, we must have $p = q$ when the curve on j is real; and the degree of the curve is even. If we take Π as the diametral plane parallel to the tangent plane at O, the projection becomes the well-known stereographic projection. The generators OA, OB become the circular lines through O in the tangent plane at O, while A and B become the circular points ω and ω' in Π. The properties of conics on j proved in § 5 now become the well-known theorems that a circle on j and the pole of its plane project into a circle and its centre, and that two circles on j whose planes are conjugate project into orthogonal circles. In particular, all great circles project into circles orthogonal to a fixed (unreal) circle which is the projection of the circle at infinity. We may show similarly that angles are unaltered by stereographic projection. These results also follow from the fact that a spherical curve and its stereographic projection from O are inverses of each other with respect to O.

If the generators through a point F on the sphere j touch a curve on j, F may be called a *focus* of the curve. The generators through F project into lines through ω and ω'. Hence the stereographic projections of the foci of a spherical curve are the foci of the projected curve.

Suppose a spherical curve of even degree n has δ nodes and κ cusps; while m great circles can be drawn through any point to touch the curve, τ great circles are bitangent to the curve, and ι great circles osculate it. Then the projected curve is of degree n and class m; it has δ nodes and κ cusps besides a $\frac{1}{2}n$-ple point at each circular point; while it has τ bitangent circles and ι osculating circles which are orthogonal to any fixed circle.

Looking at the results of Ch. XI, § 11, Ex. 1, we get

$$m = \tfrac{1}{2}n^2 - 2\delta - 3\kappa, \quad \iota = \tfrac{3}{2}n(n-2) - 6\delta - 8\kappa,$$
$$n = m(m-1) - 2\tau - 3\iota.$$

These may be regarded as Plücker's equations for a spherical curve.

Ex. 1. Two points P, P' on a sphere are said to be 'inverse with respect to a circle' on the sphere, if the line PP' passes through the pole of the plane of the circle. If P traces out a locus on the sphere, so does P', and the loci of P and P' are 'inverse' with respect to the circle. Show that (i) the stereographic projection of a circle and two inverse points is a circle and two inverse points. (ii) The angle between two spherical curves is equal to the angle between their inverses with respect to any circle of the sphere. (iii) The inverse of a spherical $2n$-ic is in general a spherical $2n$-ic of the same type (with the same Plucker's numbers). (iv) A self-inverse spherical $2n$-ic is the intersection of the sphere with a cone of degree n.

[For (i) use the fact that the circle is cut orthogonally by any circle through the points. It follows that the stereographic projection of a circle and two inverse curves on the sphere is a circle and two inverse curves; whence we get (ii) and (iii).]

Ex. 2. The foci of a spherical 4-ic lie by fours on four circles with respect to which the 4-ic is self-inverse.

The 4-ic is the intersection of the sphere with four cones of the second degree.

Ex. 3. Obtain properties of the spherical 4-ic from those of the plane bicircular 4-ic.

Ex. 4. Show that properties of a plane bicircular quartic whose real foci are concyclic may be obtained from the properties of a conic as follows: Project the conic on to a sphere from the centre. Then project the spherical curve thus obtained (a sphero-conic) stereographically into a bicircular quartic.

Ex. 5. A bicircular quartic has real concyclic foci A, B, C, D. Show that the bitangents to the curve from the intersection O of AB and CD make equal angles with OAB, OCD.

[The tangent-arcs to a sphero-conic from a point P make equal angles with the focal distances of P. Project from the point diametrically opposite to P. See also Ch. XVIII, § 3, Ex. 5.]

Ex. 6. A chord PQ of a circular cubic subtends a right angle at a fixed point O of the curve. Show that the circle through PQ bisecting the circumference of a fixed circle with centre O passes through two fixed points.

Ex. 7. Two bicircular quartics with the same four real concyclic foci cut orthogonally.

[Two confocal sphero-conics cut orthogonally. See also Ch. XVIII, § 6, Ex. 6.]

Ex. 8. P and Q are two points on a bicircular quartic with real concyclic foci A, B, C, D. Show that the four circles APC, AQC, BPD, BQD are all touched by the same two circles.

Ex. 9. The locus of a point on the Earth's surface at which a given place has a given azimuth is a spherical quartic.

Ex. 10. The spherical quartic has a node O and foci S, S'. Show that, if P is any point on the curve,

$$(\sin \tfrac{1}{2}SP \,.\, \operatorname{cosec} \tfrac{1}{2}OS \pm \sin \tfrac{1}{2}S'P \,.\, \operatorname{cosec} \tfrac{1}{2}OS') \operatorname{cosec} \tfrac{1}{2}OP$$

is constant, the plus or minus sign being taken according as O is an acnode or crunode.

[Projecting stereographically from O we project the curve into a conic.]

Ex. 11. Two spherical quartics with a common node and foci cut orthogonally.

Ex. 12. Two orthogonal circles are drawn through the node O of a spherical quartic touching the curve. The locus of their second intersection is a circle.

Ex. 13. Two circles are drawn through the cusp O of a spherical quartic touching the curve and cutting each other at a constant angle. The locus of their second intersection is a spherical quartic with a node at O.

[Projecting stereographically from O we have: 'The isoptic locus of a parabola is a hyperbola.']

Ex. 14. A spherical quartic has a node O and foci S, S'. A circle through O touching the curve is met again by the orthogonal circles through O and S, O and S' at Y, Y'. Show that Y and Y' lie on a fixed circle, and that

$$\sin \tfrac{1}{2}SY \,.\, \sin \tfrac{1}{2}S'Y' \,.\, \operatorname{cosec} \tfrac{1}{2}OY \,.\, \operatorname{cosec} \tfrac{1}{2}OY'$$

is constant.

[The reader will find a very interesting discussion of spherical quartics in Darboux's *Sur une classe remarquable de courbes et de surfaces algébriques* (Paris, 1873), pp. 1-60.]

Ex. 15. A spherical sextic has a triple point O. Show that three real circles of curvature of the sextic pass through O, and that their points of contact lie on a circle through O.

[Projecting stereographically from O we have: 'A cubic has three real collinear inflexions.']

Ex. 16. Show that the method of Ex. 4 is applicable to any curve which is self-inverse with respect to a circle.

Ex. 17. Two properties of a plane curve are derived by projecting a given spherical curve stereographically from two different points of the sphere. Show that the two properties are obtainable from each other by ordinary inversion.

Ex. 18. Through a given point O of a sphere any great circle OP is drawn meeting a given spherical $2n$-ic in $Q_1, Q_2, \ldots, Q_{2n}$. If

$$0 = \Sigma \sin \tfrac{1}{2}OQ_1 \,.\, \sin \tfrac{1}{2}OQ_2 \,.\, \ldots \,.\, \sin \tfrac{1}{2}OQ_r \,.$$
$$\operatorname{cosec} \tfrac{1}{2}PQ_1 \,.\, \operatorname{cosec} \tfrac{1}{2}PQ_2 \,.\, \ldots \,.\, \operatorname{cosec} \tfrac{1}{2}PQ_r$$

(the summation extending over ${}^{2n}C_r$ terms), find the locus of P.

[Project stereographically from the point diametrically opposite to O and use Ch. VII, § 1.]

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